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APPLICATIONS

OF

PLANE AND SPHERICAL

TRIGONOMETRY.

BY

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PREFACE.

In this book nothing more has been attempted than an introduction to the subjects specified, and a logical application of the principles of Trigonometry to them; so that a student, faithfully mastering these chapters, may understandingly pursue the subjects in larger treatises.

In the preparation of the text, and in the choice of examples in the chapter on Surveying, the author acknowledges special obligation to "Gillespie's Land Surveying," from which he has received valuable hints. "Bowditch's Navigator" has been frequently consulted, both for the definitions and for the materials of the text in the chapter on Navigation. In both these chapters, however, the author has used his own methods in presenting the subjects.

Articles referred to by number are to be found in the author's "Elements of Plane and Spherical Trigonometry," already published. Other references are to "Todhunter's Euclid" and to "Chauvenet's Geometry."

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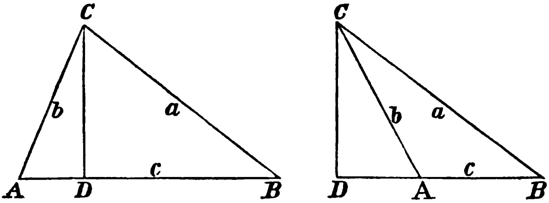
CHAPTER XVI.

APPLICATION OF PLANE TRIGONOMETRY TO SOME OF THE PROBLEMS OF MENSURATION.

ART. 165. *To find the area of a triangle, two sides and the included angle being given.*

Suppose the two sides b, c, and the included angle A of the triangle ABC are given; to find its area.



Let the perpendicular CD be drawn to the side c, or c produced.

$CD = b \sin. A$. ((1) Art. 30), (Art. 46).

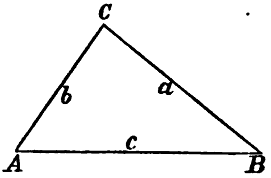
Area $= \frac{1}{2} c \times CD$ (Ch. 5, IV.),

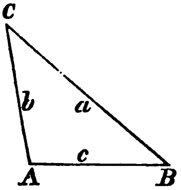
$= \frac{1}{2} bc \sin. A$.

Therefore, the area of a triangle is equal to one-half the product of any two adjacent sides multiplied by the sine of the included angle.

166. *To find the area of a triangle, of which a side and two adjacent angles are given.*

Suppose c and the angles A and B are given.





$$b = c \times \frac{\text{sin. } B}{\text{sin. } C} = \frac{c \text{ sin. } B}{\text{sin. } (A + B)} \text{ (Arts. 48 and 46);}$$

$$\text{Area} = \tfrac{1}{2}\, bc \text{ sin. } A = \frac{c^2}{2}\,\frac{\text{sin. } A \text{ sin. } B}{\text{sin. } (A + B)}.$$

167. *To find the area of a triangle, of which the three sides are given.*

a, b, and c being given, it is required to find the area of ABC.

$$\text{Let } \frac{a+b+c}{2} = s; \text{ then } \frac{b+c-a}{2} = s - a;$$

$$\frac{a-b+c}{2} = s - b; \text{ and } \frac{a+b-c}{2} = s - c.$$

$$\text{Area} = \tfrac{1}{2}\, bc \text{ sin. } A \text{ (Art. 165).}$$

$$= bc \text{ sin. } \tfrac{1}{2} A \text{ cos. } \tfrac{1}{2} A \text{ ((}a\text{) Art. 73).}$$

$$= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

((5) and (7) Art. 61).

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore, the area of a triangle is equal to the square root of the product of four factors; viz., the half sum of the sides and the three quantities obtained *by taking each one of the sides in succession from the half sum.*

EXAMPLE 1. The two sides of a triangle are 45 inches and 65 inches, and the included angle is 74° 10′. Required the area.

$$\text{Solution: Area} = \frac{45 \times 65}{2} \sin.\ 74^\circ\ 10'$$

$$= 1407.01 \text{ sq. in.}$$

2. Two angles of a triangle are 62° and 40°, and the side adjacent to these angles is 82 feet. Required the area.

$$\text{Area} = \frac{\overline{(82)}^2 \sin.\ 62^\circ \sin.\ 40^\circ}{2 \quad \sin.\ (62^\circ + 40^\circ)} = 1950.72 \text{ square feet}$$

$$\begin{aligned}
2 \log.\ 82 &= 3.827628 \\
\text{Log. sin. } 62^\circ &= 9.945935 \\
\text{Log. sin. } 40^\circ &= 9.808067 \\
\text{Ar. co. log. sin. } 102^\circ &= 0.009596 \\
\text{Ar. co. log. } 2 &= 9.698970 \\
\hline
\text{Log. } 1950.72 &= 3.290196
\end{aligned}$$

3. The three sides of a triangle are 80, 90, and 100 feet respectively. Required the area.

Solution:

$$\text{Let } a = 80 \text{ and } s - a = 55, \quad \text{Area} = \sqrt{135 \times 55 \times 45 \times 35}$$

$$\begin{array}{ll}
b = 90 & s - b = 45 \\
c = 100 & s - c = 35 \\
\hline
\text{Then } s = 135, & 3s - 2s = 135
\end{array}
\qquad
\begin{aligned}
\text{Log. } 135 &= 2.130334 \\
\text{Log. } 55 &= 1.740363 \\
\text{Log. } 45 &= 1.653213 \\
\text{Log. } 35 &= 1.544068 \\
2 \,|\, &\overline{7.067978}
\end{aligned}$$

$$\text{Area} = 3419.71 \text{ square feet.} \qquad \text{Log. } 3419.71 = 3.533989$$

4. Find the area of a triangle, of which two sides are 75 feet and 125 feet, and the included angle 115°. *Ans.* 4248.31 sq. ft.

5. Find the area of a triangle, of which two sides are 4 feet 2 inches, and 6 feet, and the included angle 125°. *Ans.* 10.24 sq. ft.

6. Find the area of a triangle, of which one side is 16 inches and the adjacent angles are 35° 10′ and 75° 16′. *Ans.* 76.086 sq. in.

7. Find the area of a triangle, of which the three sides are 348 feet, 192 feet, and 520 feet. *Ans.* 1 R., 26 sq. rd., 9 sq. yds., 6.9 sq. ft.

8. Find the area of a triangular field, of which the three sides are 41.2 chains, 33 chains, and 56.2 chains. *Ans.* 673.409 sq. ch.

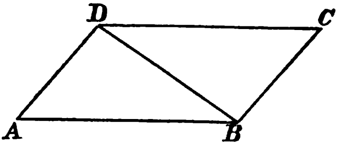
9. Find the area of a triangular field, of which the three sides are 21.6 chains, 14.1 chains, and 31.42 chains. ***Ans.*** **129.287 sq. ch.**

10. Find the area of an equilateral triangle, of which one side is 42 feet. ***Ans.*** **763.83 sq. ft.**

11. Find the area of an equilateral triangle, of which one side is 5 feet 4 inches. ***Ans.*** **12 sq. ft. 45.6 sq. in.**

168. *To find the area of a parallelogram.*

Let $ABCD$ be a parallelogram. Draw the diagonal BD. Denote the sides, AB and AD by a and b.



$ABCD = 2ABD$ (Euc. 34, I. Ch. Art. 105, I.);
$= ab$ sin. A (Art. 165).

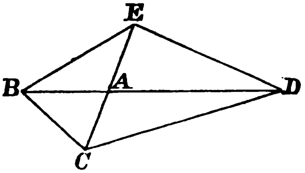
Therefore, to find the area of a parallelogram; *measure any two adjacent sides and the included angle; multiply together the two sides and the sine of the included angle, and the product will be the area.*

169. *To find the area of any quadrilateral.*

Let $BCDE$ be any quadrilateral.

Draw the diagonals, BD and CE.

Denote the angle BAC by A.



$BCDE = ABC + ACD + ADE + ABE.$
Area $ABC = \frac{1}{2} AB \times AC$ sin. A (Art. 165).
Area $ACD = \frac{1}{2} AC \times AD$ sin. A (Arts. 165, 46).
Area $AED = \frac{1}{2} AD \times AE$ sin. A.
Area $ABE = \frac{1}{2} AE \times AB$ sin. A.
$\therefore$ Area $BCDE = \frac{1}{2} \{(AB + AD)\, AC + (AD + AB)\, AE\}$ sin. A.

$$\text{Area } BCDE = \tfrac{1}{2}\,\{BD\,(AC + AE)\}\text{ sin. } A.$$
$$= \tfrac{1}{2}\,BD \times EC \text{ sin. } A.$$

Therefore, to find the area of any quadrilateral, *measure the diagonals and the angle which the diagonals make with each other; multiply the product of the diagonals by the sine of the angle, and divide the result by two.*

EXAMPLE 1. Two adjacent sides of a parallelogram are 15 inches and 18 inches, and the included angle is 35°. Required the area.

Ans. 15 × 18 sin. 35° = 154.8655 sq. in.

2. Two adjacent sides of a parallelogram are 2 feet 3 inches and 1 foot 8 inches, and the included angle is 55° 10′. Required the area.

Ans. 3 sq. ft. 11.24 sq. in.

3. The two diagonals of a quadrilateral are 16 feet and 30 feet, and the angle, which they make with one another, is 40°. Required the area.

Ans. 154.269 sq. ft.

4. The two diagonals of a quadrilateral field are found to be 18.26 chains and 15.61 chains, and the angle, which they make with one another, is 75° 15′. Required the area. *Ans.* 137.823 sq. ch.

5. One of the angles of a rhombus is 60°, and the diagonal, drawn from this angle to the angle opposite, is 20 inches. Required the area.

Ans. 115.47 in.

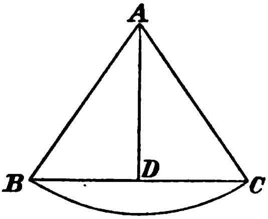
6. One of the sides of a rhombus is 16 inches, and one of the angles is 56°. Required the area. *Ans.* 212.234 sq. in.

170. *To find the area of any regular polygon, each of whose sides is equal to the unit of length.*

The *number* of the sides of the polygon will vary according as the polygon is a pentagon, hexagon, and so on. Let n denote this number; n will, therefore, represent 5 in the pentagon, 6 in the hexagon, etc.

Suppose a circle to be described about the polygon (Ch. 2. V.). Then the polygon can be divided into n equal triangles, by drawing radii to the angular points of the polygon.

Each of the angles at the centre, subtended by a side, is equal to $\frac{360°}{n}$. (Ch. Art. 7. V.)



Suppose A to represent one of these equal angles, subtended by the side BC, which, by hypothesis, is 1. From A let fall the perpendicular, AD, to BC.

AD bisects BC (Euc. 3, III., Ch. 7, II.), and, therefore, bisects the angle BAC (Euc. 4, I., Ch. 20, I.).

AD is the apothem of the polygon (Ch. Art. 6, V.);

$$BD = \frac{BC}{2} = \tfrac{1}{2};$$

$$AD = BD \text{ tan. } B \text{ ((2) Art. 32)} = \tfrac{1}{2} \times \text{cot. } BAD$$

$$= \tfrac{1}{2} \times \text{cot. } \frac{360°}{2n} = \tfrac{1}{2} \times \text{cot. } \frac{180°}{n}.$$

Area of the polygon is equal to $\frac{n \times BC \times AD}{2}$ (Ch. 8, V.).

$$\therefore \text{Area} = \frac{n}{2} \times \tfrac{1}{2} \times \text{cot. } \frac{180°}{n} = \frac{n}{4} \text{ cot. } \frac{180°}{n}.$$

171. *To find the area of any regular polygon, the length of one of its sides being given.*

Let n denote the *number* of its sides and l the length of one of its sides; then the

$$\text{Area} = l^2 \frac{n}{4} \text{ cot. } \frac{180°}{n}.$$

Denote by a the area of the polygon of the same number of sides, each of whose sides is 1. Let A be the area required, then—

$$\frac{A}{a}=\frac{l^2}{1} \text{ (Ch. 9, IV.)};$$

$$\therefore A = l^2 \times a = l^2 \times \frac{n}{4} \text{ cot. } \frac{180^\circ}{n}.$$

Therefore, to find the area of any regular polygon, *multiply together the square of the length of a side, the number of the sides, and the cotangent of half the angle subtended by a side, and divide the product by* 4.

EXAMPLE 1. Find the area of a regular pentagon, each of whose sides is 1. *Ans.* Area $= \frac{5}{2} \times \frac{1}{2}$ cot. $36^\circ = \frac{5}{4} \times 1.37638 = 1.720475$.

2. Find the area of a regular dodecagon, each of whose sides is 1. *Ans.* 11.19615.

3. Find the area of a regular hexagon, each of whose sides is 5 inches. *Ans.* 64.952 sq. in.

4. Find the area of a regular decagon, each of whose sides is 8 inches. *Ans.* 492.429 sq. in.

172. *To find the length of an arc of a circle whose radius is given, the number of degrees, minutes, or seconds in the arc being also given.*

Let n denote the number of degrees in the arc, or its subtended angle at the centre, n representing any number, integral or not integral.

Thus, if the arc be an arc of 4°, or subtends an angle at the centre of 4°, n would be 4; if the arc be an arc of 3″, n would be $\frac{3}{3600}$ or $\frac{1}{1200}$; if the arc be an arc of 3° 4′ 22″, n would be $3\frac{262}{3600}$ or $3\frac{131}{1800}$.

Let a be the arc whose length is required, r the radius of the circle, A the circular measure of the subtended angle at the centre.

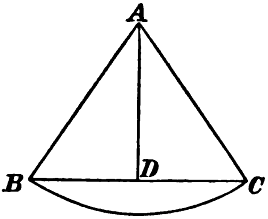
$$\frac{a}{r} = A \text{ (Art. 80)} = \frac{n\pi}{180} \text{ (Art. 83)};$$

$$\therefore a = \frac{n\pi r}{180} \text{ (Art. 83).}$$

173. *To find the length of an arc of a given radius:*

1. The *chord* being given; or,

2. The *distance of* the *chord from* the *centre* being given.



Denote the radius BA by r, the chord BC by c, and the distance, AD, of the chord from the centre, by d. AD or d bisects the chord BC, and also the angle A.

$$1.\ \frac{BD}{BA} = \frac{c}{2r} = \sin.\ BAC = \sin.\ \frac{A}{2}.$$

Thus we can find the degree measure of A; that is, of the arc BC (Ch. 20, II.). Then we can find the length of the arc, by the previous article (172).

$$2.\ \text{Again, } \frac{AD}{AB} = \frac{d}{r} = \cos.\ \frac{A}{2}.$$

From this equation we find the degree measure of the arc BC, and then the length of the arc by Art. 172.

From 1, $A = 2 \sin.^{-1} \dfrac{c}{2r}$ (Art. 76), where A denotes the degree measure of the angle A, expressed by n in Art. 172. a denoting the length of the arc BC,

$$(a)\ a = \frac{n\pi r}{180} \text{ (Art. 172)} = \frac{\pi r \sin.^{-1} \frac{c}{2r}}{90}.$$

In same manner, from 2.

$$(b)\ a = \frac{\pi r \cos.^{-1} \frac{d}{r}}{90}.$$

Example 1. Find the length of an arc of 22° 30′ in a circle whose radius is 4 inches. *Ans.* $\frac{22\frac{1}{2} \times 3.1416 \times 4}{180} = \frac{3.1416}{2} = 1.5708$ in.

2. Find the length of an arc of 5° 12′ in a circle whose radius is 2 feet 6 inches. *Ans.* 2.723 in.

3. In a circle of which the radius is 12 inches, find the length of the lesser arc, the chord being 6 inches.

$$\textit{Ans. } a = \frac{3.1416 \times 12 \times \sin.^{-1} \frac{6}{24}}{90} = \frac{3.1416 \times 12 \times 14\frac{1719}{3600}}{90}.$$

$$= \frac{6.2832 \times 14.4775}{15} = 6.064 \text{ in.}$$

4. In a circle of which the radius is 3 feet, find the length of the lesser arc subtended by a chord, of which the distance from the centre is 1 foot and 4 inches. *Ans.* 6.66 feet.

5. The diameter of the earth being nearly 7920 miles, what is the length of a degree at the equator? *Ans.* 69.115 miles.

174. *To find the area of a sector of a circle of given radius:*

1. when the *angle* of the sector is given;
2. when the *chord* of the sector is given.

Let n denote the number of degrees, minutes, or seconds in the angle of the sector, r the radius of the circle, and A the required area.

The length of the arc of the sector equals $\frac{n\pi r}{180}$ (Art. 172).

(*a*) $A = \frac{1}{2} r \times$ length of the arc (Ch. Art. 44, V.); substituting value of length of arc,

$$(1)\ \therefore A = \frac{n\pi r^2}{360}.$$

Let c denote the length of the chord of the sector.

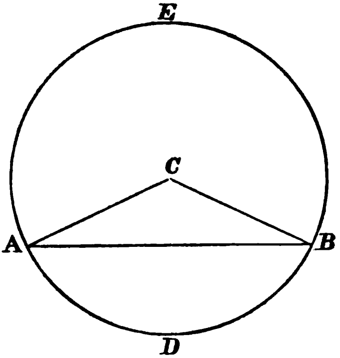
The length of the arc $= \dfrac{\pi r \sin.^{-1} \dfrac{c}{2r}}{90}$ ((*a*) Art. 173); substituting this value in (*a*)

$$(2) \therefore A = \frac{\pi r^2 \sin.^{-1} \dfrac{c}{2r}}{180}.$$

175. *To find the area of the segment of a circle of given radius:*

1. *the angle at the centre subtended by the arc being given;*

2. *the chord of the segment being given.*



Let ADB be a segment of a circle of given radius, AC. Its area is required. First, let the angle C be given.

The area of the segment is equal to the difference between the area of the sector, $ACBD$, and that of the triangle ACB.

Denote the area required by A; the radius by r; and the degree measure of C by n.

The area of the sector $ACBD$ is $\dfrac{n\pi r^2}{360}$ ((1) Art. 174).

The area of the triangle ACB is $\dfrac{r^2}{2} \sin n$ (Art. 165).

$$1. \therefore A = ACBD - ACB = \frac{r^2}{2}\left(\frac{n\pi}{180} - \sin. n\right).$$

Next, suppose the chord AB is given, and let it be denoted by c.

Then, $C = 2\sin.^{-1}\dfrac{c}{2r}$ (1 Art. 173 and Art. 76).

$$\text{Sector } ACBD = \frac{\pi r^2 \sin.^{-1}\dfrac{c}{2r}}{180}; \quad ((2) \text{ Art. } 174).$$

$$\text{Triangle } ACB = \frac{r^2}{2}\sin.\left(2\sin.^{-1}\frac{c}{2r}\right). \quad (\text{Art. } 165.)$$

$$2. \therefore A = \frac{r^2}{2}\left\{\frac{\pi\sin.^{-1}\dfrac{c}{2r}}{90} - \sin.\left(2\sin.^{-1}\frac{c}{2r}\right)\right\}$$

If it were required to find the area of the larger of the segments cut off by the chord, the triangle ACB should be added to the sector AEB.

In this case the *salient* angle ACB would be greater than 180°, and would be $(360° - ACB)$. The expression for the area in 1 would then be

$$A = \frac{r^2}{2}\left\{\frac{n\pi}{180} + \sin.(360° - n)\right\} = \frac{r^2}{2}\left\{\frac{n\pi}{180} - \sin.\, n\right\}\text{—}$$

same expression as above; but here n is $> 180°$, and therefore $\sin.\, n$ is a *minus* quantity (Art. 88).

The expression for the area in 2 would be

$$A = \frac{r^2}{2}\left\{\frac{\pi\left(180° - \sin.^{-1}\dfrac{c}{2r}\right)}{90°} + \sin.\left(2\sin.^{-1}\frac{c}{2r}\right)\right\}.$$

Example 1. Find the area of a sector of a circle, the radius being 25 inches, and the angle of the sector being 42°.

$$\textit{Ans.}\ \text{Area} = \frac{42 \times 3.1416 \times 625}{360} = 229.075 \text{ sq. in.}$$

2. Find the area of a sector of a circle, the radius being 6 inches, and the chord of the sector being 7.2 inches.

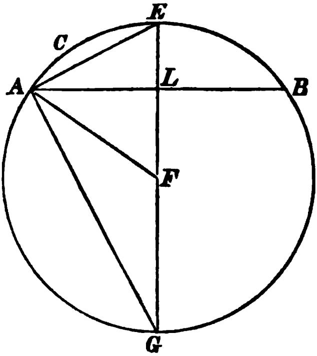
$$\textit{Ans.}\ \text{Area} = \frac{3.1416 \times 36 \sin.^{-1}(0.6)}{180}$$

$$= \frac{3.1416 \times 36.87}{5} = 23.166 \text{ sq. in.}$$

3. Find the area of a segment of a circle, of which the radius is 25 inches, the angle at the centre subtending the arc of the segment being 35°. *Ans.* Area 11.65 sq. in.

4. Find the area of the smaller segment of a circle cut off by a chord 4 inches long, the radius being 8 inches. *Ans.* 0.6796 sq. in.

Art. 176. *To find the area of a zone* of one base, the radius* of the sphere, and the *polar distance† of the base* of the zone being given.



Suppose the whole sphere to be described by the revolution of the semicircle *EAG* about the diameter *EG*. Let the arc *AE* be the arc which, by its revolution, describes a zone of one base, whose area is required. Let *AB* be the diameter of the base of the zone. Let the number of degrees in the arc *ACE* be given, and be denoted by n. Also, let the radius *FE* of the sphere be given, and be denoted by r.

Draw the chords, *AE* and *AG*. Then $AGE = \frac{1}{2} AFE$ (Euc. 20, III., Ch. 21, II.).

Area of zone described by

$$AE = \pi(AE^2) \text{ (Ch. Art. 40, IX.)};$$

$$= \pi(GE^2 \sin.^2 AGE);$$

$$= 4\pi r^2 \sin.^2 \frac{n}{2}.$$

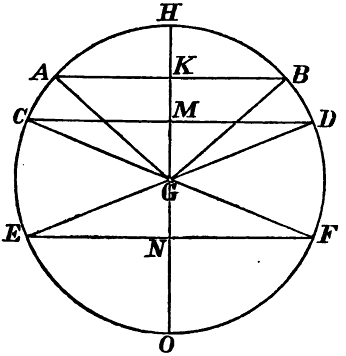
---

* Ch. Def. Art. 34, IX.

† Ch. Def. under Art. 36, VIII.

Art. 177. *To find the area of a zone of two bases, the radius* of the sphere, and the *polar distances* of the *bases* of the zone being given.

First, suppose both bases to be on the same side of the centre of the sphere. Let *HEOF* be a section of the sphere made by passing a plane through the pole (*H*), of the bases of the zones. Let *AB* and *CD* be the diameters of the bases of the zones. *GH* will cut *AB* and *CD* at right angles at points *K* and *M* (Ch. Art. 27, VIII.). Then will *HA* and *HC* be the polar distances of the bases of the zone. Denote the polar distance *HC* (or its subtended angle, *CGH*) by *m*; also, the polar distance *HA* (or its subtended angle, *AGH*) by *n*. Denote the altitude *KM* by *h*; the radius of the sphere by *r*; and the area of the zone described by the revolution of *AC* by *S*.



$$MK = h = GK - GM = r \cos. n - r \cos. m$$
$$= r(\cos. n - \cos. m)$$
$$= 2r \sin. \frac{m+n}{2} \sin. \frac{m-n}{2}$$

((*d*), Art. 70).

(1) $S = 2\pi rh$ (Ch. Art. 38, IX.)

$$= 4\pi r^2 \sin \frac{m+n}{2} \sin. \frac{m-n}{2}.$$

Next, suppose the bases to be on opposite sides of the centre, and that AB and EF be the diameters of the bases. Let HA be denoted by n, and EO by m.

Then $KN = h = GK + GN = r\,(\cos.\, n + \cos.\, m)$

$$= 2\,r\cos.\frac{m+n}{2}\cos.\frac{m-n}{2}$$

((c) Art. 70).

(2) $S = 2\pi rh = 4\pi r^2 \cos.\frac{m+n}{2}\cos.\frac{m-n}{2}$.

EXAMPLE 1. Find the area of a zone of one base, the radius of the sphere being 6 inches, and the polar distance of the base being 30°.

Ans. Area $= 4 \times 3.14169 \times 36 \times \sin.^2 15° = 30.304$ sq. in.

2. Find the area of a zone of one base, the radius of the sphere being 18 inches, and the polar distance of the base being 75°.

Ans. 10.478 sq. ft.

3. Find the area of a zone of two bases, the bases being on the same side of the centre, their polar distances being respectively 80° and 30°, and the radius of the sphere being 15 inches. *Ans.* 978.27 sq. in.

CHAPTER XVII.

APPLICATION OF PLANE TRIGONOMETRY TO PROBLEMS OF DISTANCES AND HEIGHTS.

ART. 178. The surface of the earth is a *curved* surface. In the measurement of great distances as well as in the determination of large areas on it, the curvature is taken into account. For short distances and for areas of moderate dimensions it is considered as a *plane* surface, coinciding with the *plane* of the *horizon.*

179. The *plane of the horizon* of any place is the plane tangent to the earth's surface at that place.

180. A *horizontal plane* at any place is any plane parallel to the plane of the horizon of that place. *Any two horizontal planes at the same place are parallel to each other*, as they may both be proved perpendicular to the same straight line, which is perpendicular to the plane of the horizon (Ch. Arts. 24 and 27, VI.).

181. A *horizontal line* is a straight line parallel to, or lying in, the plane of the horizon.

182. A *vertical line* is a line perpendicular to the plane of the horizon.

A vertical line at any place is perpendicular to any horizontal plane at that place (Art. 180; Ch. Art. 27, VI.).

A vertical line can be determined, in practice, by the position which a string, attached to a plummet, takes when the plummet is suspended and allowed to come to a state of rest.

183. A *vertical plane* is a plane passed through a vertical line. It is, therefore, perpendicular to the plane of the horizon, or to a horizontal plane (Ch. Art. 47, VI.).

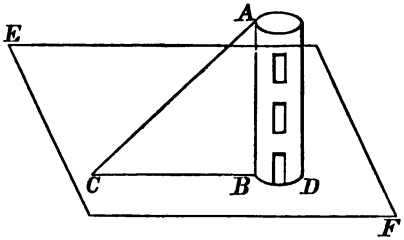
184. The *height* of a *point* above a horizontal plane is the vertical distance from the point to the plane.

185. The *height* of an *object* is the vertical distance from the top of the object to a horizontal plane passed through the base of the object.

186. If a point be *above* a horizontal plane, its *angle of elevation*, at any point in this plane, is the angle, which a straight line between the two points makes with the plane.

187. If an object stands in a horizontal plane, its *angle of elevation*, at any point in the plane, is the angle made with the plane by a straight line from the top of the object to the point.

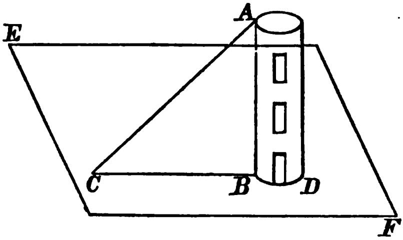
188. The angle of elevation of either a point, or an object, is an *acute* angle (Ch. Art. 60, VI.).



Suppose AD to be a tower standing in a horizontal plane, EF. Let the height of the tower be AB. At C the angle of elevation of the tower is ACB (Art. 187, Ch. Arts. 56, 57, and 59, VI.). It is included by two lines, one from the point C to the top of the object, and the other from the point C to the foot of the perpendicular (Art. 185) from A upon the plane. ABC is therefore a right-angled triangle, right angled at B (Ch. Art. 6. VI.).

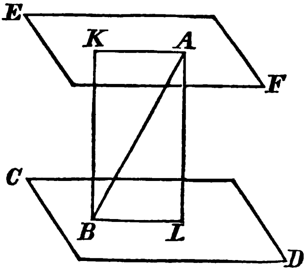
Therefore, *the angle of elevation of an object standing in a horizontal plane is the acute angle of a right-angled triangle, of which the perpendicular is the height of the object.*

189. If the point of observation be elevated, *the angle of depression* of any point *below* it is the angle, at the point of observation, made with the horizontal plane, passing through this point, by a straight line joining the two points.



Thus, if, in the figure of the preceding article, the point of observation be at A, the angle of depression of the point C, would be the angle made at A, by the straight line AC with a horizontal plane passed through A.

190. The *angle of depression*, taken at a point *above* a horizontal plane, of a point *in* the plane, *is equal to the angle of elevation*, taken at the *second* point, of the height of the *first* point above the plane.



Let A be the point above the horizontal plane, CD, and B the point in the plane. Draw the straight line AB. Through A let a horizontal plane, EF, be passed. The angle, which AB makes with EF, is equal to the angle which it makes with CD.

From A draw AL perpendicular to CD. AL is also perpendicular to EF (Ch. Art. 27, VI.). The plane ABL will be perpendicular to the two planes EF and CD (Ch. Art. 47, VI.), and will, therefore, project the line AB upon these planes (Ch. Arts. 56 and 57, VI.). Let AK be the projection of AB upon the plane EF, and BL be the projection of AB upon CD. BAK is the angle, which AB makes with the plane EF, and ABL is the angle, which AB makes with the plane CD (Ch. Art. 60, VI.).

Now, EF and CD are parallel planes (Art. 180); therefore, AK and BL are parallel lines (Ch. Art. 25, VI.). Consequently, the angles BAK and ABL are equal (Euc. 29, I., Ch. Art. 49, I.). But BAK is the angle of depression of B, at the point A (Art. 189), and ABL is the angle of elevation of A, at the point B (Art. 186). Therefore, the theorem of the article is proved.

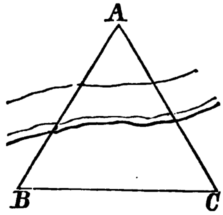
191. It follows from the preceding article that, if the point of observation be at the top of an object, which stands in a horizontal plane, *the angle of depression* of any *point*, in that plane, is equal to *the angle of elevation* of the *object*, taken *from* the *point* in the plane.

192. Angles are measured by means of various instruments. A *quadrant* measures angles of elevation and depression only, or angles in a vertical plane. A *surveyor's compass* measures horizontal angles only, or angles in a horizontal plane. A *theodolite* measures angles in both vertical and horizontal planes. A *sextant* measures angles in any plane.

193. *To find the distance of an inaccessible object.*

Suppose it is required to find the distance from B to A, A being inaccessible.

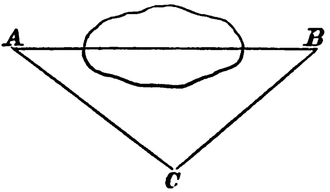
Measure from B a straight line, BC, to a point C, where the object A is visible, and where, also, the station B is visible. Measure also the angles, ABC and BCA. Then, in the triangle ABC, we have given two angles and a side in order to find another side (Art. 51).



194. *To find the distance between two objects separated by an impassable barrier.*

If the observer stationed at one object can see the other, the problem can be solved by the method of the preceding article. For, in the figure of that article, if A and B are two objects, the method of finding the distance, AB, is the same as before.

If, however, we have two objects, *A* and *B*, separated by a barrier such as to prevent the observer stationed at one from seeing the other, let a third point, *C*, be taken, such that both objects can be seen from it. Measure the distances *AC* and *CB* and the angle *C*. The distance *AB* can then be found by Art. 53.



EXAMPLE 1. An observer standing at a point on one side of a stream, and wishing to find the distance of an object on the other side, measures, from his point of observation to a second point, a straight line of one hundred yards, and finds the angle, between this line and a line drawn to the object, at the first point, to be 53° 20′, and at the second point to be 59° 30′. Required the distance from the object to each point. *Ans.* 93.489 yds.
Ans. 87.032 yds.

2. A man on the seashore, wishing to ascertain the distance of a buoy from the point where he is standing, measures, on the beach, a base-line of 90 rods in length, and at each extremity of the line takes the angle included between this line and the line to the buoy. The angle at one extremity of the base-line is 45°, the angle at the other extremity is 56° 15′. Required the distance from the buoy to each extremity of the base-line. *Ans.* 76.298 rds., and 64.886 rds.

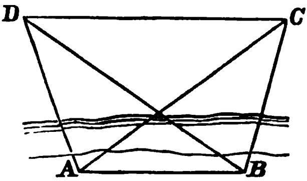
3. A man wishing to find the distance between two objects, each invisible from the other, selects a point from which both are visible. He finds the distances from this point to the objects to be 109 yards and 76 yards respectively, and the angle between these distances to be 101° 30′. Required the distance between the objects.
Ans. 144.77 yds.

4. A surveyor, wishing to find the distance across a pond, between two points at which his view is obstructed by an island, selects a third point, at which he can see the staves set up at the two first-mentioned points. The distances between the staves and the point of observation are found to be 181 and 129 rods, respectively, and the angle between these distances to be 110°. Required the distance across the pond. *Ans.* 255.68 rds.

195. *To find the distance between two inaccessible objects, which are in the same plane with the observer.*

Thus, suppose an observer to be at *A* on the shore of a bay, and that he desires to find the distance between two rocks, *C* and *D*, in the bay.

Let the distance *AB* be measured on the beach. At *A* let the angles *BAC* and *BAD*, and at *B* the angles *ABC* and *ABD*, be taken. Then, solving the triangle *ABD* (Art. 51), we find *AD* or *BD*, and, solving the triangle *ABC*, we find *BC* or *AC*. Next, we solve either the triangle *ADC*, or *BDC* (Art. 53).



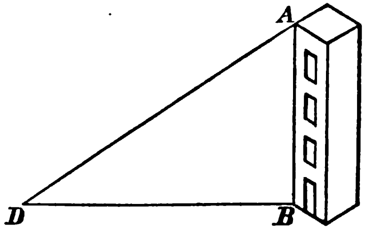
Example. Suppose, in the case just given, the distance on the beach to be 280 yards, the angle *ABC* to be 110°, *ABD* to be 52° 20′, *BAD* to be 95°, and *BAC* to be 47° 30′. Required the distance *CD*.

Ans. 509.77 yds.

196. *To find the height of an accessible object standing in a horizontal plane, the side of the object containing the height.*

Suppose it is required to find the height of a tower, of which the walls are vertical.

Measure a line from the base of the tower to some convenient point, *D*, in the plane, in which the tower stands. Take the angle of elevation at *D*. Then the height, *AB*, can be found by (2) Art. 32.



Example 1. What is the height of a vertical wall, of which the angle of elevation at a distance of 25 feet from the base of the wall is 72° 15′?

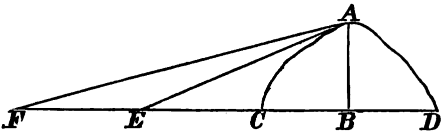
Ans. 78.1. ft.

2. Required to find the height of a vertical wall, of which the angle of elevation, 20 feet away from it, is 47° 10′. What is also the distance from the top of the wall to the point of observation?

Ans. Height = 21.57 ft. Dist. = 29.42 ft.

197. *To find the height of an accessible object standing in a horizontal plane, the side not containing the height.*

Thus, suppose it were required to find the height, *AB*, of a hill standing in a horizontal plane.



Measure in a straight line, proceeding from the foot of the hill, the distance *EF*. At *E* measure the angle *AEB*, and at *F* the angle *AFE*.

The angle *AEF* is the supplement of *AEC*. In the triangle *AEF* we can find *AE* (Art. 51), and then *AB* in the triangle *AEB* (Art. 30).

Example 1. In the preceding case, suppose the distance *EF* to be 612 yards, the angle *AEB* to be 27° 29′, and the angle *AFE* to be 15° 36′; what is the height of the hill? *Ans.* 368.84 yds.

2. The height of a tree is required, the tree standing in a horizontal plane. A point is taken in the plane, and, at that point, the angle of elevation of the tree is found to be 64°. From this point a distance of 88 feet is measured in a straight line, which, when produced, would pass through the foot of the tree. At the extremity of this line the angle of elevation is 35°. What is the height? *Ans.* 93.58 ft.

198. *To find the height of an inaccessible object which stands in a horizontal plane.*

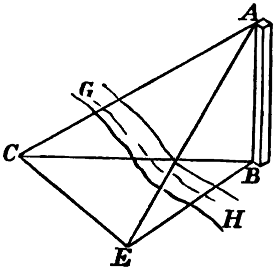
(*a*) The method of the preceding article can be used to solve the problem. For, if, in the figure of Art. 197, the observer at *E* is separated from the hill by a river, the height can be found as before.

(*b*) In case it is not convenient to measure the straight line in the direction proceeding from the foot of the object, the following method can be adopted:

Suppose *AB* to be a column standing in a horizontal plane, but not accessible to the observer, at *C*, on account of an intervening stream, *GH*.

Measure a straight line, *CE*. At *C* and *E* measure the angles, *BCE* and *CEB*. Also, at one point—for instance, at *C*—take the angle of elevation *ACB*. Then *CB* can be calculated (Art. 51), and *AB* (Art. 32).

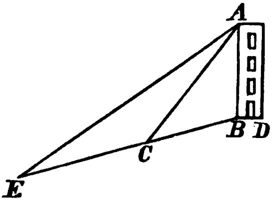
In the figure, *AB* is supposed to be an object whose side is vertical, and one, also, of which the base is visible from the stations *C* and *E*. If, however, the object has not its side vertical, or if the base were not visible from the two stations, we can still find the parts of the triangle *ECB*. For if we project the line from *C* to *A* upon the plane, and also the line from *E* to *A* upon the plane, we can measure the angles *ECB* and *CEB* as before, and solve the problem.



EXAMPLE 1. In the figure of this article, if *CE* is 212 feet, the angle *ECB* is 25°, *CEB* is 107°, and *ACB* is 15°, what is the height of *AD*? *Ans.* 73.099 ft.

2. An observer on the seashore, desiring to find the height, above the sea-level, of a lighthouse standing on a ledge of rocks, measures on the beach a base-line of 421 feet. At each extremity of this line he takes the angle made by this line with the projection of the line from each point to the top of the lighthouse. These angles are 75° and 55°. At the point where the first angle is measured the angle of elevation of the top of the lighthouse is 20°. What is the height? *Ans.* 163.85 ft.

199. *To find the height of an object which stands on an inclined plane, the object having a vertical side.*



Suppose *AD* to be a vertical tower in an inclined plane. From the foot of the tower measure two distances, *BC* and *CE*, in the same straight line. At *C* and *E* take the angle subtended by the object. Then *AC* can be found (Art. 51), and *AB* (Art. 53).

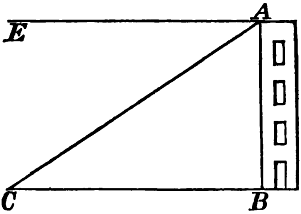
EXAMPLE 1. In the figure, suppose $BC = 16$ yards, $CE = 25$ yards, $ACB = 35° 20'$, $AEB = 20° 30'$. What is the height of the tower?

Ans. 23.082 yds.

2. Wishing to find the height of a vertical wall at the foot of a hill of even slope, the observer measured up the hill a distance of 25 feet, and then found the angle subtended by the wall to be 75°. Continuing, he measured 35 feet more, in the same straight line, when he found the angle to be 38°. What was the height of the wall? *Ans.* 37.996 ft.

200. *The observer being at the top of an object which stands in a horizontal plane, to find the distance of any object in the plane, the height of the observer's station being known.*

Suppose an observer to be at A, the top of a vertical tower of known height, and that the distance BC or AC is required.



Measure the angle of depression, EAC. As $EAC = ACB$ (Art. 190), BC or AC can be found (Art. 31).

EXAMPLE 1. In the preceding case, suppose $AB = 120$ feet, and the angle $CAE = 34° 3' 30''$. What is the distance CB? *Ans.* 177.52 ft.

2. A man on the top of a hill, 575 feet high, wishes to know his distance from a point in the plane below. He takes the angle of depression, and finds it to be $25° 31' 10''$. Required the distance from the man to the point. *Ans.* 1334.67 ft.

MISCELLANEOUS EXAMPLES.

1. A man, at a height of 1200 feet above a horizontal plane, measures the angles of depression of two objects in the straight line proceeding from the foot of the perpendicular, dropped upon the plane, from the point of observation. He finds the angles to be 50° and 30°. What is the distance between the objects? *Ans.* 1071.54 ft.

2. A man in a balloon, in order to find his altitude, at the moment he is directly over one place, takes the angle of depression of a second place, distant from the former place 3 miles. If the angle of depression is 25°, what is his distance above the first place? *Ans.* 1 mile 2106.32 ft.

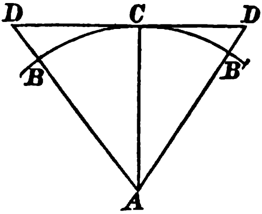
3. To find the height of a kite, directly above a line joining two stations, its angle of elevation at each station was observed. The distance between the stations was known to be 650 yards. The angle of elevation at one station was 35°, at the other the angle was 70°. Required the height of the kite. *Ans.* 362.7 yds., nearly.

4. A flag-staff, 20 feet in height, stands on the top of a building. The angle of elevation of the top of the building, taken at a point in the plane, in which the building stands, is 35°. At the same point, the angle

of elevation of the top of the staff is 40°. Required the height of the building. *Ans.* 100.88 ft.

5. At the top and base of a wall, 22.4 feet high, the angles subtended by an elevated object, standing in the same horizontal plane with the wall, are found to be 38° and 35° respectively. Required the height of the object. Show that, with the given parts, there may be two solutions. *Ans.* 32.198 ft. or 73.632 ft.

6. *Required to find the distance of an object of known height from the observer, the top of the object being just visible on the horizon.*



Solution: Let DB be the height of the object. First. Suppose the eye of the observer to be at C, on the surface of the earth.

Let DB produced pass through the centre, A, of the earth. Join A and C by the straight line AC.

As D is on the horizon, DC is a tangent line (Arts. 179 and 181, and Ch. Art. 45, VIII.). Let $DB = h$; $DC = d$; and $AB = R$.

$$\begin{aligned} DC^2 &= DA^2 - AC^2; \\ d^2 &= (R + h)^2 - R^2; \\ &= 2Rh + h^2; \\ d &= \sqrt{2Rh + h^2}. \end{aligned}$$

$2R$ is so large, in general, compared with h, that h^2, except for great heights, may be neglected, and the formula becomes

$$d = \sqrt{2Rh}.$$

Second. Suppose the eye of the observer is elevated, as at D'; we find CD' in the same way. Calling CD' d', and BD' h', we have $d' = \sqrt{2Rh' + h'^2}$, and, where h' is small, $d' = \sqrt{2Rh'}$.

In this second case, DD' is found by adding d and d'.

7.* At what distance can the top of a hill, 486 feet in height, be just seen, supposing the eye of the observer is at the sea-level, and that the diameter of the earth is 7920 miles?

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* In this example and in the following examples no allowance is made for refraction of light by the earth's atmosphere. The formulas of the preceding example are only correct on the hypothesis that light moves in straight lines.

$$\text{Solution: } d = \sqrt{\frac{(7920) \times 486}{5280}} = \sqrt{729} = 27 \text{ miles.}$$

8. The light of a lighthouse is 550 feet above the sea-level. At what distance can it be just seen on the horizon? *Ans.* 28.7 miles.

9. The light of a lighthouse is 135 feet above sea-level. At what distance can it be seen by an observer, whose eyes are 40 feet above sea-level? *Ans.* 21.97 miles.

10. The distance from the observer of a light, just visible on the horizon, is known to be 30 miles. Supposing the eyes of the observer to be at the sea-level, what will be the height of the light above sea-level?

$$\text{Solution: } h = \frac{d^2}{2R} = \frac{30^2}{7920} \text{ miles} = \frac{900 \times 5280}{7920} \text{ ft.} = 600 \text{ ft.}$$

11. Suppose the top of a hill is just visible, and the distance from it is known to be 50 miles. Required its height. *Ans.* 1666.7 ft.

CHAPTER XVIII.

APPLICATION OF PLANE TRIGONOMETRY TO SURVEYING.

ART. 201. *Surveying* is the art of determining the areas of portions of the earth's surface.

The areas considered in this chapter are areas of moderate dimensions, and are, therefore, treated as *plane* areas, and the lines inclosing these areas are considered as straight lines (Art. 178).*

202. The *meridian* of a point, or place, is a line running north and south through the point, or place.

Meridians of different points are considered as parallel.

203. The *parallel* of *latitude* of a point, or place, is a line running east and west through the point or place.

Parallels of latitude of different points are parallel to each other.

204. The *meridian* and the *parallel* of *latitude* of a point are, in this chapter, considered as *straight* lines *perpendicular* to each other.

205. The latitude of a point is its distance north or south of some fixed parallel of latitude, and is denoted north or south latitude according as the point is north

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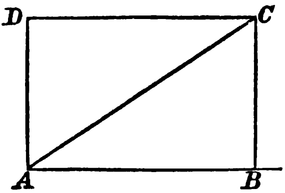
*** All the lines mentioned in these definitions are *really* curved lines, being lines on the surface of the earth.**

or south of the fixed parallel of latitude. This distance is measured on the meridian of the point.

206. The *difference* of latitude of two points is the distance between the parallels of latitude of the two points:

(1.) The difference of latitude of two ends of a *line* is, therefore, the distance between the parallels of latitude of the extremities of the line.

(2.) This difference of latitude of the two ends of the line is called *the* latitude of the line, and is frequently spoken of as the *northing* or *southing* of the line, because it is measured on a north and south line.



Thus, in the figure, suppose DC and AB to be parallels of latitude of the extremities, A and C, of the straight line AC. Then, if AD, a meridian line, is drawn, it will be perpendicular to CD (Art. 204) and AB (Art. 203). AD is the northing of the line AC, if we regard the north point to be represented at the upper part of the paper.*

If the line AC be read in the other direction, as the line CA, CB, the perpendicular upon AB, would be the southing of CA.

From the figure it is evident (Euc. 34, I., Ch. 30, I.) that the northing of a line, read in one direction, is equal to its southing when read in the opposite direction.

207. The longitude of a point is its distance east or west of some fixed meridian.

208. The *difference of longitude* of two points is the distance between the meridians of the two points:

(1.) The difference of longitude of the two ends of a *line* is, therefore, the distance between the meridians of the extremities of the line.

---

* North and south lines are represented on paper as running in one direction—as, for instance, up and down the paper; and east and west lines as at right angles to north and south lines (Art. 204).

(2.) This difference of longitude of the two ends of a line is called the *departure* of the line, or its easting or westing, because this distance is measured on an east and west line.

If AC be a line, and parallels of latitude, AB and CD, and meridians, AD and BC, are drawn through its extremities, the *departure* of AC would be represented by AB, which in this case would be an *easting*, as the east point is generally represented as at the right-hand side of the paper.

If the line were read from C to A, as CA, then CD would represent its *departure*, which in this case would be a *westing*, as the distance is measured to the left, and the west point is generally represented at the left-hand side of the paper.

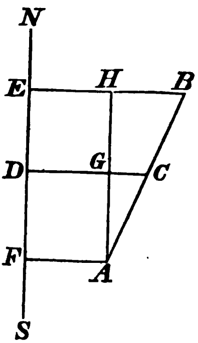
Also, the easting of a line read in one direction is equal to its westing when the line is read in the opposite direction.

209. The *longitude* of a *straight line* is the distance of the *middle* point of that line from some fixed meridian.

The longitude of a line is, also, often spoken of as its *meridian distance.*

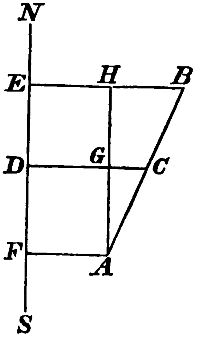
If AB be a given straight line, and NS be a given meridian, the longitude of AB would be represented by the straight line CD, drawn from C, the middle point of AB, perpendicular to NS.

If the meridian to which the longitude of AB is referred passed, as AH, through one extremity of the line, CG would be the longitude of AB.



210. (1.) If, through the extremities of the line AB, parallels of latitude BE and AF are drawn, they will be perpendicular to the fixed meridian NS. These lines BE and

AF are together double *CD* (Ch. Arts. 123 and 124, V.). But *BE* and *AF* are the longitudes of the extremities of the line; therefore, *the longitudes of the extremities of a line are together double the longitude of the line*, or equal the *double longitude of the line*.



(2.) If the meridian passed through *A*, one extremity of the line, *BH* would be its departure ((2) Art. 208), and would be double of *CG*, its longitude; therefore, if the *meridian passes* through *one* extremity of a line, the *departure* of the line is equal to the *double longitude*.

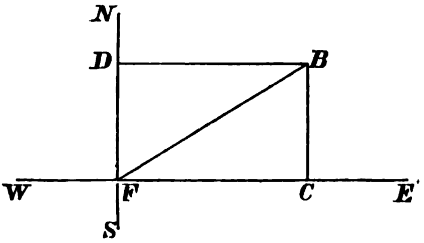
Also, $EB - FA = EB - EH = HB$; but $EB - FA$ is the difference of the longitudes of the two points *B* and *A*, and *HB* is the departure of the line *AB*; therefore, the *difference of longitude of the two extremities of a line is equal to the departure of the line*, a principle which has been assumed in the definitions under Art. 208.

211. The *bearing* of a line, at any point, is the *angle* which the line makes with the meridian (or north and south line) through that point. If the line runs east and west, the bearing is a right angle (Art. 204). In all other cases the bearing is an acute angle.

Thus, draw two straight lines, *NS* and *WE*, at right angles to one another at the point *F*; *N* and *S* representing north and south, and *E* and *W* representing east and west, respectively. Through *F* draw any line, *FB*. The *bearing* of *FB*, at the point *F*, is the angle *DFB*.

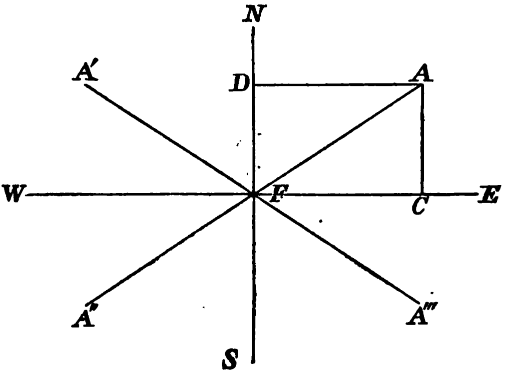
212. If, through *B*, the meridian *BC*, and the parallel of latitude *BD*, be drawn, then the figure *DC* will be a rectangle (Arts. 202–204).

The angle *FBC* = angle *BFD* (Euc. 29, I., Ch. Art. 49, I.).



FBC is called the *reverse bearing* of *FB*. Therefore, *the bearing* and *reverse bearing* of a line are equal, the bearing being the angle made by the line with the meridian of one extremity of the line, and the *reverse* bearing being the angle made by the line with the meridian of the other extremity.

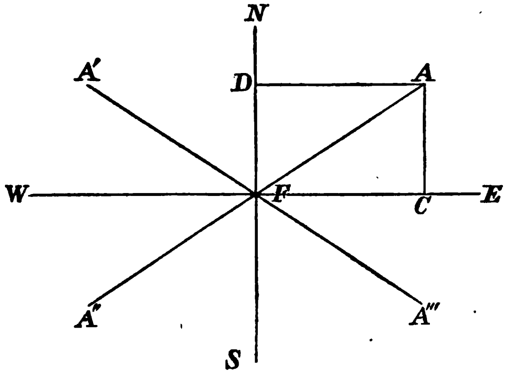
213. Suppose through the point *F* two lines, *NS* and *WE*, be drawn, at right angles to each other, *N*, *S*,



E, *W* representing respectively the north, south, east, and west points of the compass.

If a line, as *AF*, be drawn from *F* in the angle

NFE, the line will run in *some* direction between north and east, its *particular direction* being determined by its bearing, *DFA*. If the bearing alone were given—



that is, the acute angle which the line makes with *NS* —the line would not be determined in *position*, for four lines can be drawn from *F* equal to each other, one in each of the angles *NFE*, *NFW*, *SFW*, *SFE*, and each making the same acute angle with *NS*.

In order to determine the position of a line in any one of these four angles it is customary to write the degree measure of the bearing between the letters denoting the two main points of the compass, between which the line falls, *N.* or *S.* being always written *before* the degree measure, and *E.* or *W.* *after* the degree measure.

Thus, suppose the lines *AF*, *A'F*, *A''F*, *A'''F*, each makes with *NS* an angle of 40°. Then the bearing of *FA* is N. 40° E.; of *FA'* is N. 40° W.; of *FA''* is S. 40° W.; of *FA'''* is S. 40° E. These expressions are read North 40° East, North 40° West, etc.

A line running east or west, north or south, is said to have a bearing E. or W., N. or S., and needs no degree measure to determine its position.

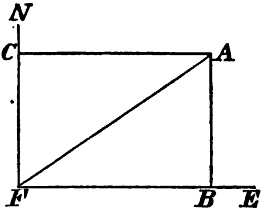
214. *Signs of Lines.*—All lines running *east* will be considered *positive*, and all lines going *west* will be considered *negative.* Also, all *north* lines will be considered *positive*, and all south lines *negative.*

Of course, these signs are *arbitrarily* given, simply to distinguish the *opposite* directions of the lines. We might agree to consider lines running *west* positive, and then lines running *east* would be negative. So, also, if we considered lines running *south* as positive, lines running *north* would be negative.

Therefore, in the figure of the preceding article, FC will be positive, and FD will be positive. But FC is the *departure* ((2) 208) of FA, and FD is the *latitude* ((2) 206) of FA. Consequently, FA has a *plus* departure and a *plus* latitude. In the same manner $A'F$ may be shown to have a *plus latitude* but a *minus departure*, and $A''F$ to have a *minus latitude* and a *minus departure;* $A'''F$ to have a *minus latitude* but a *plus departure.*

215. *To calculate the latitude and departure of a line from its length and bearing.*

Let FN be a meridian and and let FE be a parallel of latitude through F, one extremity of a line, FA, of which the length and the bearing, AFC, are given.



Through A draw the meridian, AB, and the parallel of latitude, AC. Then CB is a rectangle (Arts. 202–204).

FB is the departure, and FC the latitude of FA.

$$FB = CA = FA \text{ sin. } AFC \text{ ((1) Art. 30)},$$
$$FC = FA \text{ cos. } AFC \text{ ((2) Art. 30)};$$

that is, departure = distance (or length) × sine of bearing,
latitude = dist. × cos. bearing.

As the length (or distance), bearing, departure, and latitude, may be represented by FA, AFC, CA ($= FB$), and FC respectively, in the right-angled triangle AFC, the relations between these quantities are such as exist between the four parts, considered in the solution of a right-angled triangle (Arts. 24–32).

216. *Gunter's chain* is the measure commonly used in surveying. It is four rods in length, and is divided into one hundred links.

1 chain = 4 rds. = 66 ft. = 100 links.
1 sq. chain = 16 sq. rds. = 10,000 sq. links.
1 acre = 160 sq. rds. = 10 sq. chs. = 100,000 sq. links.

As the chain contains a *hundred* links, chains and links can be written together as whole numbers and decimals. For instance, 5 chains and 35 links are written 5.35, the unit being the chain.

The dimensions of a field being given in chains and links, the area will be expressed in square chains and square links.

(*a*) To reduce the area to acres, it will be seen, from the table, that we must divide the number of square chains by 10, or, we must remove the decimal point one place to the left. The remaining figures to the right will be decimals of an acre, which can be reduced to roods and perches (or square rods) by the usual methods.

For instance, suppose the area of a field to be

84.0355 sq. chains. Removing the decimal point,
this equals 8.40355 A.
4
1.61420 R.
40
24.568 sq. rds. or perches.

The area is, therefore, 8 A., 1 R., 24.568 P.

217. In order to calculate the area of a field, the length of each side is measured, and its bearing is taken with reference to a meridian, drawn through one extremity. Each angular point is called a *station*, and the length of each side is called a *distance*.

Any angular point may be taken as the *first* station. The length of the side first measured from this point is called the *first distance*. At the extremity of this distance, remote from the first station, a *second station* is taken and a second distance is measured, and so on until all the sides of the field are measured.

In the examples of this chapter the first station will be generally taken as the extreme western angular point.

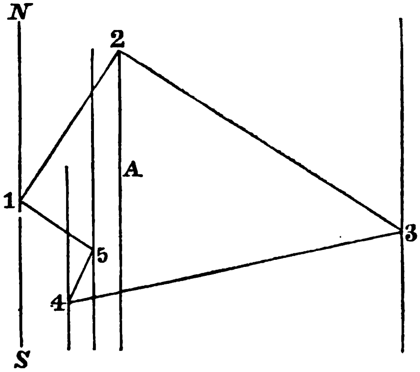
218. *To plot a field when the distances and their bearings are given.*

Suppose a field is to be plotted, of which the following table contains the distances and their bearings:

Stations.	Bearings.	Distances.
1	N. 33° 30′ E.	3.18 chains.
2	S. 57° E.	5.77 chains.
3	S. 78° 8′ W.	6.03 chains.
4	N. 33° 15′ E.	1.05 chains.
5	N. 56° 30′ W.	1.54 chains.

Take any point, 1, to represent the first station, and through 1 draw *NS* to represent a meridian. At the point 1 make the angle *N*1 2 equal to 33½° (by means of a line of chords, or by means of a protractor—see Chap. I.). From a plane scale measure off 1 2, equal to 3.18 units. Through 2 draw a second meridian, 2*A*, and at 2 make the angle *A* 2 3 equal to 57°. From 2 measure 2 3, equal to 5.77 units. At 3, 4, and 5 proceed to complete the plot in the same manner, by drawing meridians and making the bearings and distances as given in the table.

From 5 it is not *necessary* to lay off the last bearing, but it is well to do so, in order to test the accuracy of the plot, or survey; for, if the line, which makes, with the meridian through 5, the last bearing equal to the one given, does not pass through 1 or very near 1, there must be some error in the plot or survey.



The same unit must be used to lay off all the distances.

219. *To calculate the area of a field, by the method of latitudes and departures.*

Suppose it is required to find the area of the field, of which the notes and the plot are given in the preceding article.

Through 1, 2, 3, 4, and 5 draw $1T$, $B2C$, $3AD$, $P4K$, $5MF$, perpendicular to NS, the meridian of 1. These lines will also be perpendicular to the meridians $2A$, CP, $4M$, and $5L$ (Arts. 202–204), and will form, with these meridians, the parallelograms BA, AC, etc.

With reference to the meridian NS,

(*a*) $B2$ is the double longitude of 1 2 ((2) Art. 210);

(*b*) $B2+D3=$ double longitude of 2 3 ((1) Art. 210);

(*c*) $D3+K4=$ double longitude of 3 4;

(*d*) $K4+F5=$ double longitude of 4 5;

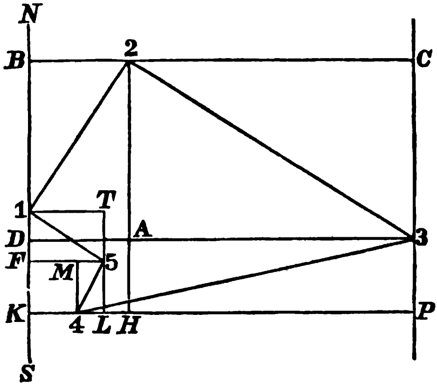
(*e*) $F5=T1=$ double longitude of 5 1.

Also,

(1.) $1B$ is the latitude, or northing of 1 2 ((1) and (2) Art. 206);

(2.) $2A$ is the latitude, or southing of 2 3;

(3.) $3P$ is the latitude, or southing of 3 4;



(4.) $4M$ is the latitude, or northing of 4 5;

(5.) $F1=5T=$ the latitude, or northing of 5 1.

A. The area 1 2 3 4 5 $=$ (area $B23D$ + area $D34K$) $-$ (area $1B2$ + area $K45F$ + area $F51$) $=\frac{1}{2}\{(2\times$ area $B23D$ $+\ 2\times$ area $D34K)-(2\times$ area $1B2+2\times$ area $K45F+$ $2\times$ area $F51)\}$.

Now, area $B23D=\dfrac{B2+D3}{2}\times 2A$ (Ch. Art. 17, IV.).

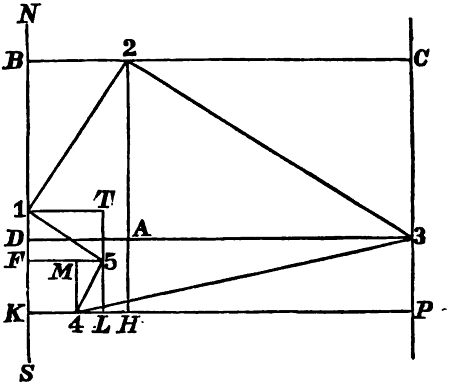
$\therefore\ 2\times$ area $B23D=(B2+D3)\times 2A$;

$=$ doub. long. $\times$ lat., of line 23 ((*b*) and (2)).

In the same way it may be shown that

$2 \times \text{area } D34K = (D3 + K4) \times 3P =$ doub. long. × lat. of line 34 ((*c*) and (3));

$2 \times \text{area } K45F = (K4 + F5) \times 4M =$ doub. long. × lat. of line 45 ((*d*) and (4)).



Again, area $1B2 = \dfrac{B2 \times 1B}{2}$ (Ch. Art. 13, IV.).

$\therefore 2 \times \text{area } 1B2 = B2 \times 1B =$ doub. long. × lat. of 12 ((*a*) and (1)).

Also, $2 \times \text{area } F51 = F5 \times F1 =$ doub. long. × lat. of 51 ((*e*) and (5)).

B. The areas $2(1B2)$, $2(K45F)$, $2(F51)$ are called *north areas*, or *north products*, as they are each obtained from factors, one of which is a *northing*. Also, the areas $2(B23D)$ and $2(D34K)$ are called *south areas* or *south products*, as they are each obtained from factors, one of which is a *southing*. Therefore, from A. and B.

C. Area $12345 = \frac{1}{2}$ the difference between *north* products and *south* products.

Also, in A., substituting for 2 × area $B23D$, 2 × area $D34K$, etc., their equivalents, we have

D. Area 1 2 3 4 5 = $\frac{1}{2}$\{(doub. long. × lat. of 2 3 + doub. long. × lat. of 3 4) − (doub. long. × lat. of 1 2 + doub. long. × lat. of 4 5 + doub. long. × lat. of 51)\}.

From D it is seen that the area of a field can be expressed in terms of the double longitudes, and the latitudes of the sides of the field. The *latitudes* can be calculated from the lengths and bearings of the sides (Art. 215). We can also find the *departures* from the lengths and bearings of the sides (Art. 215). It only remains to show that the *double longitudes* can be found from the *departures.*

E. The *double longitude* of the *first side is its departure.*

$B2 =$ doub. long. of 12 ((2) Art. 210).

The *double longitude* of the *second* side is equal to the sum of the *double longitude* of the *first* side, the *departure* of the *first* side, and the *departure* of the *second* side.

(The sum being the result of the *algebraic* addition of these quantities, E. departures being *plus*, and W. departures being *minus.*)

Double longitude of 23 is $B2 + D3$;

$B2 + D3 = B2 + DA + A3 = B2 + B2 + 2C.$

The *double longitude* of *any* side, after the first, is equal to the *algebraic sum* of the *double longitude* of the *preceding* side, the *departure* of *that* side, and the *departure* of the *given* side.

Double longitude of 34 is $D3 + K4$:

$$\begin{aligned} D3 + K4 &= D3 + KP - P4; \\ &= D3 + KH + HP - P4; \\ &= (D3 + B2) + 2C - P4. \end{aligned}$$

Double longitude of 45 is $K4+F5$:

$$\begin{aligned} K4+F5 &= K4+KL; \\ &= K4+KP-P4+4L; \\ &= (K4+D3)-P4+4L. \end{aligned}$$

The *double longitude* of the *last* side is equal to its *departure.*

The double longitude of 51 is $F5$:

$$\begin{aligned} F5 &= (K4+F5)+M5-T1; \\ &= FM+F5+M5-F5. \end{aligned}$$

If, now, the results of this article be applied to finding the area of the field, of which the notes are given in Art. 218, it is best to arrange the notes along with the latitudes, departures, double longitudes, north products, and south products in parallel columns, as is done in the following table .

STA-TIONS.	Bearings.	Dis-tances.	Latitudes.		Departures.		Double longi-tudes.	North products.	South products.
			N.+	S.−	E.+	W.−			
1	N. 33° 30′ E.	3.18	2.65		1.76		1.76	4.6640	
2	S. 57° E.	5.77		3.14	4.84		8.36		26.2504
3	S. 78° 3′ W.	6.03		1.24		5.90	7.30		9.0520
4	N. 33° 15′ E.	1.05	.88		.58		1.98	1.7424	
5	N. 56° 30′ W.	1.54	.85			1.28	1.28	1.0880	
			4.38	4.38	7.18	7.18		7.4944	35.3024

Double longitude
of 1st side = +1.76
+1.76
+4.84
of 2d side = +8.36
+4.84
−5.90
of 3d side = +7.30
−5.90
+0.58
of 4th side = +1.98
+0.58
−1.28
of 5th side = +1.28

Double longitude of last side = its departure, 1.28 = 1.28.

35.3024
7.4944
2 | 27.8080
A. 1.3904
4
R. 1.5616
40
P. 22.464

Area = 1 A., 1 R., 22.464 P.

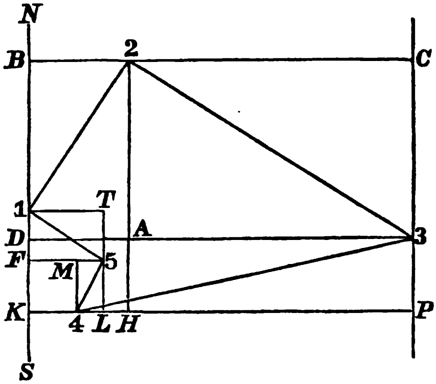
Remark.—In calculating latitudes and departures, as in measuring the sides of a field, *results* are entered only to the *nearest link;* that is, if the latitude or departure is found in chains, links, and decimals of a link, *one* is *added* to the number of the links, if the decimals equal or exceed $\frac{5}{10}$ link; in other cases the decimals are rejected.

For instance, if a latitude or departure is found to be 7.346 chains, it should be entered in the column as 7.35; but if it be found to be 7.344 chains, it should be entered 7.34.

In ascertaining the double longitudes of the sides, the *last double longitude* should *equal* the *last departure* (see E). We thus have a test of the accuracy of the column of double longitudes. If, therefore, the last double longitude is *not* equal to the last departure, some mistake has been made in forming the column. The work should be gone over till the mistake is found, and the result conforms to the test.

220. *To balance a survey.*

If a field has been correctly surveyed, and the latitudes and departures of the sides correctly calculated, the northings should *balance* the southings and the eastings should *balance* the westings; that is, the sum of the northings should *equal* the sum of the southings, and the sum of the eastings should *equal* the sum of the westings.



For, in the figure of the preceding article, the sum of the northings is $1B + 4M + 5T = 1B + KF + F1 = KB$; the sum of the southings is $2A + 3P = C3 + 3P = KB$ (Euc. 34, I., Ch. 30, I.).

In the preceding example the sum of the N. latitudes and the sum of S. latitudes each equals 4.38.

Also, the sum of the eastings is $B2 + 2C + 4L = BC + 4L = KP + 4L$.

The sum of the westings is $P4 + 5F = P4 + MF + 5M = P4 + 4K + 4L = KP + 4L$.

In the preceding example the sum of the E. departures and the sum of the W. departures each equals 7.18.

In every survey it is therefore necessary, after calculating the latitudes and departures, to compare the sum of the northings with the sum of the southings, and the sum of the eastings with the sum of the westings. If the northings *balance* the southings and the eastings *balance* the westings, the calculation of the area can be continued according to E and D, Art. 219. If the northings and southings do *not* balance, or if the eastings and westings do *not* balance, and these quantities are found to be correctly calculated from the given sides and their bearings, either the field must be resurveyed or corrections must be applied to the latitudes and departures, so as to make the N. latitudes equal the S. latitudes, and the E. departures equal the W. departures.

As to the limit of error allowed before resurveying, authorities differ.* Small errors are unavoidable. The method of correcting allowable errors will be considered and applied to an example.

The process, by which, in the calculation of the area of a field, corrections are applied, so as to make the latitudes balance and the departures balance, is called *bal-*

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* See Gillespie's "Land Surveying," p. 176.

ancing the survey. The corrections to be applied are obtained by the following proportions:

(a) $\dfrac{\text{Sum of lengths of sides}}{\text{Length of given side}} = \dfrac{\text{whole error in latitude}}{\text{correction for latitude of given side.}}$

(b) $\dfrac{\text{Sum of lengths of sides}}{\text{Length of given side}} = \dfrac{\text{whole error in departure}}{\text{correction for departure of given side.}}$

Whole error in latitude is the difference between the sum of N. latitudes and sum of S. latitudes. *Whole error in departure* is the difference between the sum of E. departures and the sum of W. departures.

The correction is to be *added* when the latitude or departure is in the column of which the sum is the *less* of the two sums, but is to be *subtracted* when the latitude or departure is in a column of which the sum is the *greater* of the two.

The rule and its application will be best understood by means of the example given below:

STATIONS.	Bearings.	Distances.	Latitudes.		Departures.		Corrected Lat.		Corrected Dep.	
			N. +	S. −	E. +	W. −	N. +	S. −	E. +	W. −
1	N. 55° 1′ E.	32.66	18.72	.....	26.76	.....	18.70	.....	26.77	.....
2	S. 25° E.	21.90	.....	19.85	9.26	.....	.....	19.86	9.26	.....
3	S. 58° 30′ W.	29.24	.....	15.28	.....	24.93	.....	15.29	.....	24.93
4	N. 34° W.	19.85	16.46	.....	.....	11.10	16.45	.....	.....	11.10
		103.65	35.18	35.13	36.02	36.03	35.15	35.15	36.03	36.03
			Error .05.		Error .01.					

$\dfrac{103.65}{32.66} = \dfrac{.05}{.016}$ ∴ .02 to be *subtracted* from 18.72.

$\dfrac{103.65}{21.90} = \dfrac{.05}{.01}$ ∴ .01 to be *added* to 19.85.

$\dfrac{103.65}{29.24} = \dfrac{.05}{.014}$ ∴ .01 to be *added* to 15.28.

$\dfrac{103.65}{19.85} = \dfrac{.05}{.009}$ ∴ .01 to be subtracted from 16.46.

The fourth terms are each found by dividing the product of the second and third terms by the first.

The error in departure being only 1 link, and our calculations not extending beyond the link (Remark, page 219), the departures are balanced by adding 1 link to the departure of the first side or distance, as that is the *longest* side.

The *double longitudes* are obtained from the *corrected departures*, and the north and south products are obtained by multiplying the double longitudes by the *corrected latitudes.*

The calculation is accordingly completed in the table on page 251.

221. The results of Arts. 219 and 220 may be summed up in the rule,

To calculate the area of a field:

1. *Find the latitude and departure of each side* (Art. 215).

2. *Balance the survey* (Art. 220), if the sum of the north latitudes does *not* equal the sum of the south latitudes, or if the sum of the east departures does *not* equal the sum of the west departures.

3. *Form the column of double longitudes* (E., Art. 219) from the *corrected* departures.*

4. *Multiply the double longitude of each side by the corrected* latitude of the side;* the product will be a north product or south product according as the latitude is north or south.

5. *Divide the difference between the north products and the south products by 2, and reduce the quotient to acres, roods, and perches* ((*a*) 216).

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* On the supposition that the survey requires balancing; otherwise use the simple departures and latitudes.

STATIONS.	Bearings.	Distances.	Latitudes.		Departures.		Corrected Lat.		Corrected Dep.		Double Longitudes.	North Products.	South Products.
			N. +	S. −	E. +	W. −	N. +	S. −	E. +	W. −			
1	N. 55° 1′ E.	32.66	18.72	.....	26.76	.....	18.70	.....	26.77	.....	26.77	500.5990	....
2	S. 25° E.	21.90	.....	19.85	9.26	.....	.....	19.86	9.26	.....	62.80	....	1247.2080
3	S. 58° 30′ W.	29.24	.....	15.28	.....	24.93	.....	15.29	.....	24.93	47.13	....	720.6177
4	N. 84° W.	19.85	16.46	.....	.....	11.10	16.45	.....	.....	11.10	11.10	182.5950	....
		103.65	35.18	35.13	36.02	36.03	35.15	35.15	36.03	36.03		683.1940	1967.8257
			Error .05.		Error .01.								683.1940

2 | 1284.6317

A. 64.231585

4

.926340

40

37.0536

Area = 64 A., 0 R., 37.054 P.

Give the plots and calculate the areas of the fields, of which the notes are—

1.

1	N. 52° E.	10.64
2	S. 29° 45′ E.	4.09
3	S. 31° 45′ W.	7.68
4	N. 61° W.	7.24

Ans. 4 A., 3 R., 28.14 P.

2.

1	N. 83° 15′ E.	5.86
2	N. 12° E.	8.25
3	N. 47° W.	4.24
4	S. 21° W.	12.41

Ans. 4 A., 2 R., 35.67 P.

3.

1	N. 35° E.	4.40
2	N. 63° 30′ E.	2.58
3	S. 57° E.	4.44
4	S. 84° 15′ W.	7.10
5	N. 56° 30′ W.	6.46

Ans. 4 A., 1 R., 19.7 P.

4.

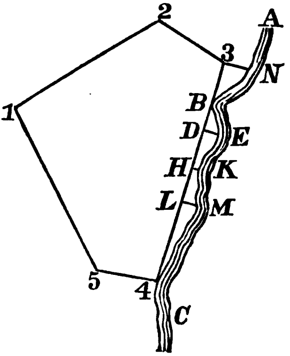
1	N. 84° E.	10.
2	N. 21° 15′ E.	17.62
3	S. 65° 30′ E.	4.98
4	S. 58° E.	8.56
5	S. 14° 15′ W.	20.69
6	S. 47° W.	0.60
7	S. 57° 15′ W.	8.98
8	N. 56° W.	12.98

Ans. 41 A., 3 R., 17.+ P.

5.

1	N. 84° E.	9.76
2	N. 67° E.	2.80
3	N. 23° E.	7.08
4	N. 18° 30′ E.	4.48
5	S. 76° 30′ E.	12.41
6	S. 21° 15′ W.	17.62
7	S. 84° W.	10.
8	N. 56° W.	14.15

If a field, as 1 2 3 N 4 5, is not bounded on all sides by straight lines, but has one side, as N B 4, adjacent to a stream, it is customary to survey the field by running a line as near as possible to this irregular boundary, and then measuring perpendiculars (called *offsets*) to this line 34, viz., 3*N*, *DE*, *HK*, and *LM*. The area 1 2 3 4 5 is first calculated, and then the areas of the figures 3*BN*, *BDE*, *DEKH*, *HKLM*, *LM*4. These last are considered as triangles and trapezoids. The total area is the sum of the areas of 1 2 3 4 5, 3*BN*, *BDE*, *DEKH*, *HKML*, *LM*4.



In case of an obstacle to the measurement of one side of a field, or in case a field has been once correctly surveyed and the length and bearing of one side have been lost from the notes, the length and bearing of the side may be obtained from the lengths of the other sides and their bearings. The difference between the eastings and westings of the known sides will be the *departure* of the remaining side, and the difference between the northings and southings of the known sides will be the latitude of the remaining side (Art. 220). From the latitude and departure we can calculate the distance and bearing required (Art. 215), and then continue the calculation of the area of the field.

Example 6. Calculate the length and bearing of the first side of a field, of which the other sides are given in the following notes. Calculate also the area of the field.

1	Unknown.	Unknown.
2	S. 21° 15′ W.	17.62
3	S. 34° W.	10.
4	N. 56° W.	14.15

Ans. Length = 29.06
Bearing = 54° 40′ 48″
Area = 21 A., 0 R., 27.398 P

CHAPTER XIX.

NAVIGATION.—PLANE SAILING.—MERCATOR'S SAILING.

In navigation the *earth* is regarded as a *sphere.* Small parts of its surface (as in surveying) are considered as *planes.*

Art. 222. The *axis* of the earth is the *diameter* about which it revolves. The extremities of this axis are called *poles*, one being named the North Pole and the other the South Pole.

223. The *meridian* of any point, or place, on the earth is the *great* circle arc passing through the point, or place, and through the poles of the earth.

The meridian of a point or place may be said to be the intersection of a plane with the surface of the earth, the plane being determined by the axis and the point (Ch. Art. 4, VI., and Art. 28, VIII.).

(*a*) Meridians are, therefore, *north* and *south* lines.

224. The earth's *equator* is the circumference of the great circle, whose plane is *perpendicular* to the axis.

(*a*) The equator is perpendicular, therefore, to the meridians (Ch. Art. 58, VIII.).

225. *Parallels* of latitude on the earth are circumferences of *small* circles, whose planes are *perpendicular* to the *axis.*

The planes of these parallels are parallel to each other and to the plane of the equator (Ch. Art 24, VI.).

(*a*) *Parallels* of *latitude* are *east* and *west* lines.

226. The *longitude* of a point, or place, is the *angle* between the plane of the meridian of the point, or place, and the plane of some fixed meridian. This angle is measured by the arc of the equator intercepted between these planes, since this arc measures the plane angle of the diedral angle of the planes (Ch. Art. 57, VIII.). This *arc*, intercepted between the two meridians, is spoken of as *the longitude*, as its degree measure is the same as that of the diedral angle.

One assumed meridian from which longitude is reckoned is the meridian of the Observatory of Greenwich, England; another is the meridian of the Observatory of Washington. The French, also, have a fixed meridian from which longitude is reckoned.

(*a*) Longitude is reckoned, on the *arc* of the equator, east and west of the assumed meridian, from 0° to 180°.

(*b*) The *difference of longitude* of two places is the angle between the planes of their meridians, and is measured by the arc of the equator intercepted between these meridians.

This arc is evidently the difference of the two arcs, which measure the longitudes of the two places, if the places are either both E. or both W. of the assumed meridian.

(*c*) If we give to E. longitudes the sign +, and to W. longitudes the sign —, the arc which measures the difference of longitude of two places will always be the *algebraic* difference of the longitudes of the places.

(*d*) To find, then, the *difference* of longitude of two places whose longitude is given, we *subtract* the *less from the greater if both are E. or both are W., but add the two if one is E. and the other W.*

227. The *latitude* of a point, or place, is the *angle* made with the plane of the equator by a line drawn from this point, or place, to the centre of the earth. The latitude is measured by the arc of the meridian (of the point) which subtends the angle. This subtending *arc* is spoken of as *the latitude*, as its degree measure is the same as that of the inclination of the line to the plane of the equator.

Latitude is reckoned from 0° to 90°, north and south of the equator.

228. The *difference of latitude* of two places is the difference between the latitudes of the two places, difference being understood as *algebraic*, and north latitudes having the sign + and south latitudes the sign —.

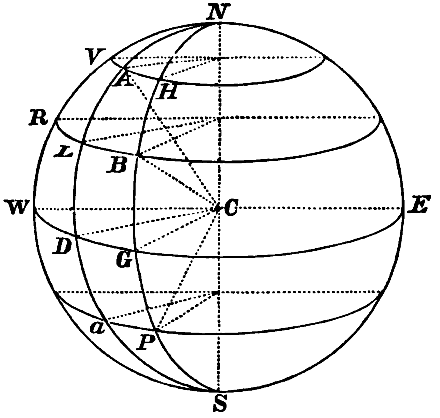
(*a*) To find, then, the *difference of latitude* of two places whose latitude is given, *take the less from the greater if both are N. or both are S., but add the two if one is N. and the other S.*

(*b*) The *difference of latitude* of two places is measured on *any arc of a meridian intercepted between the parallels of latitude of the places.*

Let the figure represent a hemisphere of the earth. Let N and S be the poles; C the centre; and WDE the equator. Suppose A and B to be two points on the surface; VAH to be the parallel of latitude of A, and NAS to be its meridian; RLB to be the parallel of latitude, and NBS the meridian of B. Then it is to be proved that the difference of latitude of A and B is measured by AL, HB, VR, or any other meridian arc intercepted between VAH and RLB.

Let the meridian NAS intersect the parallel RLB in the point L, and the equator in the point *D*. Also, let the meridian *NBS* intersect the parallel *VAH* in the point *H*, and the equator in the point *G*. Draw the straight lines *CA*, *CD*, *CB*, and *CG*.

The plane of the meridian *NAS* is perpendicular to the plane of the equator (Arts. 223 and 224, and Ch. Art. 47, VI.), and is, therefore, the plane which projects the line *CA* upon that plane (Ch. Art. 57, VI.). *CD* is the intersection of these two planes (Ch. Art. 5, VI.), and contains (as a part of it) the projection of the line *CA*. The angle *ACD* is, therefore, the angle made by the line *AC* with the plane of the equator (Ch. Arts. 59 and 60, VI.), and is, consequently, the latitude of the point *A* (Art. 227). Also, the plane of the meridian, *NGS*, is perpendicular to the plane of the equator, and by its intersection with that plane determines the projection of the line *CB* upon the plane. Therefore, *BCG* is the latitude of the point *B*.



Now, *ACD*, or the latitude of *A*, is measured by *AD*, and *BCG* is measured by *BG*; therefore, the difference of latitude of *A* and *B* is measured by the difference between *AD* and *BG*; that is, by $AD - BG$.

$AD = ND - NA = NG - NH = NW - NV$ (Ch. Art. 35, VIII.).

Also, $BG = ND - NL = NG - NB = NW - NR$.

$\therefore AD - BG = NL - NA = NB - NH = NR - NV$;

$= AL = HB = VR$, etc.

If the point *P* be taken on a parallel of latitude south of the equator, the difference of latitude would be measured by *HP* or by *Aa*, an arc of a meridian intercepted between the parallels.

229. It is evident that *the position of any point or place* on the earth's surface *is determined if the latitude and longitude of the point or place are known.*

Thus, suppose *NWSE* to represent a hemisphere of the earth; *NWS* to be the meridian from which longitude is reckoned; *WDE* to be the arc of the equator; *RLB* and *VAH* to be parallels of latitude; and *NDS*, *NBS* to be meridians.

Suppose the latitude of the point to be 30° N., and the longitude to be 40° E. If, now, *RLB* be a parallel of latitude, of which the polar distance *NR*, *NL*, or *NB* is 60° (Ch. Art. 36, VIII.), since *NW*, *ND*, or *NG* is 90° (Ch. Art. 37, VIII.), *WR*, *DL*, or *GB* is 30°; therefore, *RLB* is a parallel of *latitude*, *every point* of which is 30° N. of the equator. Consequently, the point whose latitude is given must be found somewhere on this arc *RLB*. Again, if *NDS* be a meridian, whose plane *NDS* makes with the plane *WNS* an angle of 40° (measured by the arc *WD* of the equator) the *longitude* of every point on *NDS* is 40° E. Therefore, the point whose longitude is given must be somewhere on the meridian *NDS*. Since the point is on the arc *RLB*, and at the same time on the arc *NDS*, it must be at their intersection, *L*. Therefore, the point is *determined* when its latitude and longitude are given.

It might be said that two circles intersect twice (Euc., 10, III., Ch. Art. 22, II.), and therefore that the point of the circle *RLB* diametrically opposite to *L* would be indicated by lat. 30° N., long. 40° E. This is evidently false, since the other half of *NDS* and the other half of *RLB*, which, by their intersection, determine this second point, are on the other hemisphere. The *latitude* of this second point *is* 30° N., but its *longitude* is 140° *W*. of the assumed meridian ((*a*) Art. 226).

230. As charts of the earth's surface are constructed for the use of navigators with meridians and parallels of latitude either drawn on them, or indicated, if a ship's *latitude* and *longitude* are known, the position of the ship is determined. It is important that this position should be determined from day to day, and therefore it is important that the ship's latitude and longitude should be known. Latitude and longitude are best obtained by observations of the heavenly bodies. This is a department of navigation which belongs to astronomy. It is necessary to have other methods of determining a ship's position when it is impossible to resort

to the methods of astronomy. These other methods are now to be considered.

231. (*a*) The *distance* sailed by a ship in, going from one point to another, is the *length* of the line traversed by the ship between the two points.

(*b*) The *bearing* or *course* of a ship, at any point, is the *angle* which the line traversed by the ship (that is, the *distance*) makes with the meridian passing through that point.

If a ship cuts every meridian at the *same angle*, she is said to continue on the *same course*.

If a ship is said to sail a given distance on a given course, it is assumed that in that distance she continues on the *same course*.

The path made by a ship continuing on the same course is called a *rhumb line*, or simply a *rhumb*.

(*c*) The *departure* of a ship, in sailing from one point to another, is the whole east or west distance she makes measured from the meridian from which she sails, and is an easting or westing according as she sails in an easterly or westerly direction.

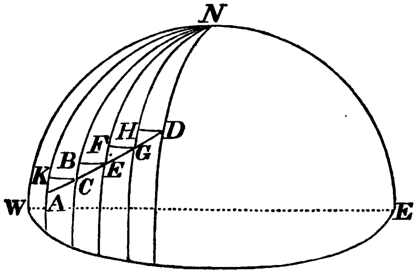
If the distance sailed is *small*, it may be considered a straight line, and the departure might also be regarded as a *straight* line measuring the perpendicular distance between the meridians of the two points. In this case the *meridians* may be considered parallel straight lines (as in surveying), since they are lines on a small portion of the earth's surface (Art. 178), and are perpendicular to the same line.

If the distance is *not* small, it may be divided up into such a number of small parts that each of them may be considered as a straight line. The departure of each of these small distances will then also be a *straight* line, and the departure of the *whole* distance will be the sum of the departures of the parts.

(*d*) *Difference of latitude* of two points has already been defined (Art. 228).

If a given distance sailed by a ship is small, it may be regarded as a straight line, and then the difference of latitude of the two extremities of the line, representing this distance, is measured by a line, which may be also regarded as a straight line. The difference of latitude is then a northing or southing (as in surveying).

If the distance is *not* small, it may be divided into such a number of parts that each part may be small enough to be considered a straight line. The difference of latitude of each part will then be a straight line, and the difference of latitude of the whole distance will be the sum of the differences of latitude of the parts.

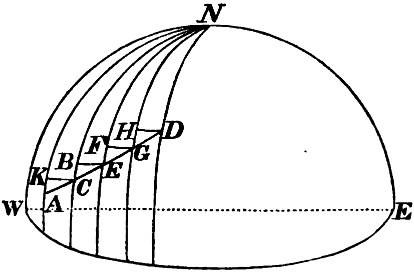


Thus, suppose AC to be a small distance on the earth's surface. Let AK and BC be meridians of the points A and C, and let these lines be considered parallel. If CK be a perpendicular to AK drawn from C, it will be the departure of AC, and AK will be the difference of latitude of A and C, or the difference of latitude for the distance AC.

If the distance be a *long* distance, as from A to D, then it can be divided into such a number of short distances—as, for instance, AC, CE,

EG, and GD—that each one of them can be considered as a straight line. If AK, CB, EF, and GH be the meridians of the points A, C, E, and G, and if CK be the perpendicular from C to AK, EB be perpendicular to CB, GF to EF, and DH to GH, then the *departure* for AD will be $KC + BE + FG + HD$, and the *difference of latitude* will be $AK + CB + EF + GH$.

232. *Plane sailing* is the art of determining the position of a ship at sea by means of a right-angled plane triangle. Of this triangle the hypotenuse is the distance, the base is the difference of latitude, the perpendicular is the departure, and the angle between the base and the hypotenuse is the course.



When the distance sailed is short, it is evident from the figure that the four quantities mentioned are the parts of a right-angled triangle; for then AC is the distance, AK is the difference of latitude, KC at right angles to AK is the departure, and CAK is the course.

If the distance sailed is not short—as, for instance, the distance AD—then divide it into such a number of small distances, AC, CE, EG, and GD, that each may be considered a straight line. Complete the figure as in the preceding article. Suppose the ship's course to be the same in sailing from A to D, then the angles CAK, ECB, GEF, and DGH are equal ((*b*) Art. 231).

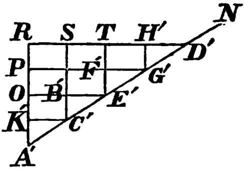
Now, take any straight line $A'N$, and on it lay off $A'C'$, $C'E'$, $E'G'$, and $G'D'$, equal respectively to AC, CE, EG, and GD, and on these lines $A'C'$, $C'E'$, $E'G'$, and $G'D'$ construct right-angled triangles $A'K'C'$,

$C'B'E'$, $E'F'G'$, and $G'H'D'$ equal respectively to AKC, CBE, EFG, and GHD, then $A'D' = AD$, the distance, and

$A'K' + C'B' + E'F' + G'H' = AK + CB + EF + GH =$ difference of latitude;

$K'C' + B'E' + F'G' + H'D' = KC + BE + FG + HD =$ departure.

Since the ship sails on the same course, the angles $K'A'C'$, $B'C'E'$, $F'E'G'$, and $H'G'D'$ are all equal, and, therefore, the lines $A'K'$, $C'B'$, $E'F'$, $G'H'$ are parallel; also, the lines $K'C'$, $B'E'$, $F'G'$, and $H'D'$ are parallel (Euc., 28, I., Ch. Art. 55, I.). Produce $A'K'$ and $D'H'$ to meet at R; produce $C'B'$ and $E'F'$ to meet $D'R$ at S and T; and produce $E'B'$ and $G'F'$ to meet $A'R$ at O and P. R is a right angle, since it is equal to K' (Euc., 29, I., Ch. Art. 51, I.). $A'RD'$ is consequently a right-angled triangle. $A'D'$ represents distance sailed. $D'A'R$ represents the course. $A'R$ represents the distance of latitude, for



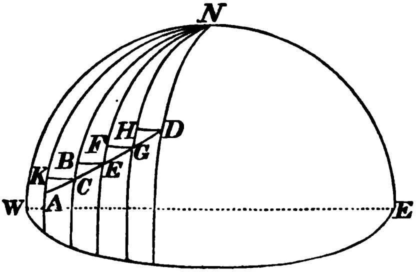
$A'R = A'K' + K'O + OP + PR = A'K' + C'B' + E'F' + G'H'$ (Euc., 34, I., Ch. Art. 104, I.).

RD' represents the departure, for

$RD' = RS + ST + TH' + H'D' = K'C' + B'E' + F'G' + H'D'$.

233. Any two parts of a right-angled triangle being given, in addition to the right angle, the other parts may be found (Arts. 24–32); therefore, of the four quantities, the *distance*, the *course*, the *departure*, and the *difference* of latitude, any two being given, the other two may be found, since these quantities may be represented by the parts of a right-angled triangle, as has been shown in the preceding article, and will, therefore, have the same relation to one another as the corresponding parts of the right-angled triangle.

When the distance is small, this is evident. If the distance is great, it may be divided, as before, into such a number of small distances, AC, CE, EG, and GD, that each may be considered a straight line. Let the differences of latitude for these small distances be AK, CB, EF, and GH, and let the departures be KC, BE, FG, and HD. As the course is supposed to be the same for the whole distance AD, the angles CAK, ECB, GEF, and DGH are all equal.



In the right-angled triangle AKC, $\frac{AK}{AC} = \cos.\ CAK = \cos.$ course;

In the right-angled triangle CBE, $\frac{CB}{CE} = \cos.\ BCE = \cos.$ course;

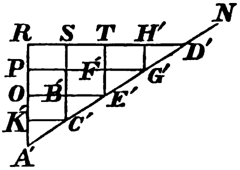
In the right-angled triangle EFG, $\frac{EF}{EG} = \cos.\ FEG = \cos.$ course; and

In the right-angled triangle GHD, $\frac{GH}{GD} = \cos.\ HGD = \cos.$ course.

Therefore (Ch. Art. 12, III.),

$$(1)\quad \frac{AK + CB + EF + GH}{AC + CE + EG + GD} = \cos.\ \text{course}.$$

Now, if we construct the right-angled triangle $A'RD'$, as in the preceding article, having $A'D' = AD$, then, as in Art. 232, it may be shown that $A'R = AK + CB + EF + GH =$ dif. of latitude, and $A'D' = AC + CE + EG + GD = AD =$ dist.



Substituting these values in (1) we have:

$$\frac{A'R}{A'D'} = \frac{\text{difference of latitude}}{\text{distance}} = \cos.\ \text{course; or,}$$

(2) difference of latitude = dist. × cos. course.

In same manner it can be shown

(3) departure = distance × sine of course;

(4) departure = difference of latitude × tan. course;

and that the other relations shown to hold between the parts of a right-angled plane triangle hold between the quantities in navigation represented by these parts.

234. If at a given time it is required to find the position of a ship by *plane sailing*, the rate of speed per hour at which she is sailing is first ascertained. This rate, multiplied by the number of hours elapsed since the last ascertained position, will give the *distance* from that position. The angle made by the direction in which the ship is headed, and the N. and S. line of the mariner's compass (with correction, if necessary), will furnish the *course*. From these data the *difference of latitude* and the *departure* are found ((2) and (3), Art. 233), and thus the position of the ship is known.

For example, suppose the average rate of sailing is ascertained to be 9 miles an hour, and that 12 hours have elapsed since the last ascertained position, then the distance is 108. If the course is observed to be N. 30° E., the ship's position N. of her last position will be, in miles, 108 × cos. 30°, or 93.5 miles, and her position E. will be 108 × sin. 30°, or 54 miles.

235. The rate of sailing is ascertained by means of the *log and line*.

The *log*, in one of its simplest forms, is a triangular piece of wood, so weighted as to assume, when attached to its line and placed in water, a position calculated to oppose the most resistance to force applied to the line. The *line* is a rope knotted at regular intervals.

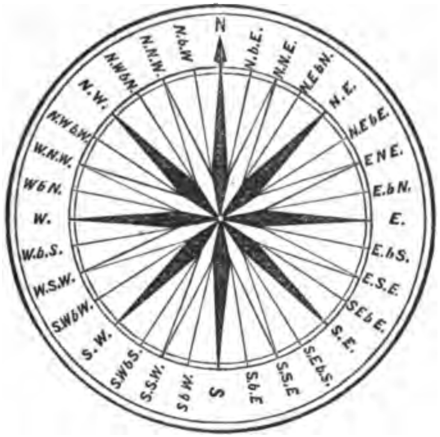
When the log is thrown overboard and the line is reeled out by the forward motion of the ship, the number of knots passing over a given point in a given period of time will give the rate of sailing for that period of time. Moreover, *if the interval between the knots be the same part of a mile that the period of time is of an hour, the number of knots passed out during the period of time will give the number of miles per hour* sailed by the ship.

For instance, let the period of time be $\frac{1}{2}$ minute or $\frac{1}{120}$ hour, then the interval between the knots must be $\frac{1}{120}$ mile. Suppose, then, 4 such knots (counting the intervals by the knots) should be reeled out by the forward motion of the ship during ½ minute, we should find the distance sailed per hour (that is, the rate per hour) by the proportion

$\frac{1}{2}$ min. : 60 min. :: $\frac{4}{120}$ mile : x (the distance per hour),

$\therefore x = 2 \times 60 \times \frac{4}{120} = 4$ miles, the same number of miles per hour as knots per half minute.

236. The *mariner's compass* consists of a circular card attached to a magnetic needle, which generally points N. and S.* Each quadrant of this card is divided into eight equal parts, called points, to which names are given as represented in the accompanying figure.†



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* The magnetic needle does not at all places on the earth's surface point N. and S. Charts for the use of navigators, however, give the amount of variation for places where the needle is subject to variation, so that for such places a correction can be applied to the direction indicated by the needle, so as to obtain a true N. and S. line.

† The naming of the points in each quadrant will be seen to be not

Also, to express courses between the points, the points are subdivided into half-points and quarter-points.

The points are read (taking the quadrant between the N. and E. points), North by East, North North East, North East by North, North East, etc., etc.

The angle between two adjacent points is $\frac{90}{8}^\circ$, or 11° 15′.

237. Distance, departure, and difference of latitude are all expressed in *nautical miles.*

A *nautical mile* is equal to a minute of an arc of the circumference of a great circle of the earth.

As there are 69.115 *common* miles in a degree of such an arc (Ex. 5, Art. 173), a nautical mile is *longer* than the common statute mile.

Differences of latitude expressed in degrees and minutes is, therefore, easily converted into miles, or, when expressed in nautical miles, is easily changed into degrees and minutes.

Thus, 5° 33′ difference of latitude = 333 miles; and 656 miles difference of latitude = 10° 56′.

EXAMPLE 1. A ship sails N. E. b. N. a distance of 70 miles. Required her departure and difference of latitude at the end of that distance. The

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without method. Thus, in the quadrant between N. and E., the point midway between N. and E. takes its name from both these points; then the point midway between N. and N. E, and the point midway between E. and N. E., take their names respectively from the two points between which each is situated, as one of these is North and the other is East of N. E.

The remaining points are named from the nearest *main* point (calling N., E., and N. E. main points), with the addition of N. or E. as the point to be named is north or east of this nearest point, with the word *by* placed between the two. Thus, the point between N. and N. N. E. is N. *by* E.; the point between N. N. E. and N. E. is N. E. *by* N.; the point between N. E. and E. N. E. is N. E. *by* E.; and the point between E. N. E. and E. is E. *by* N.

The points of the other quadrants may be shown to be named on the same method.

course is 3 points from N. toward E., and is, therefore, 3 × (11° 15') or 33° 45'. ***Ans.*** **Dep. = 38.89 miles; dif. lat. = 58.2 miles.**

2. A ship from lat. 33° 5' N. sails S. S. W. 362 miles. Required her departure and the latitude arrived at.

Ans. **Dep. = 138.5 miles; lat. 27° 30.6' N.**

3. A ship, leaving port in lat. 42° N., sails S. 37° W. till her departure is 62 miles. Required the distance sailed and the latitude arrived at. ***Ans.*** **Dist. = 103 miles; lat. 40° 38' N.**

4. A ship sails S. 50° E. from lat. 7° N. to lat. 4° S. Required her distance and departure.

Ans. **Dist. = 1026.78 miles; dep. = 786.56 miles.**

5. A ship sails from the equator on a course between S. and W. to lat. 5° 52' S., when her departure is found to be 260 miles. Required her course and the distance sailed.

Ans. **Course = S. 36° 27' W.; dist. = 437.6 miles.**

6. A ship sails from lat. 3° 2' N. on a course between N. and W. a distance of 382 miles, when her departure is found to be 150 miles. Required her course and the latitude arrived at.

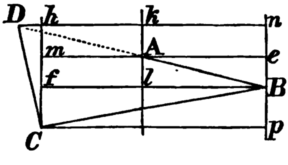
Ans. **Course = N. 23° 7¼' W.; lat. 8° 53' N.**

Art. 238. A *traverse* is the path described by a ship which changes its course from time to time.

The object of *traverse sailing* is to find the position of a ship at the end of a traverse; the *distance* sailed from the position left to the position reached; and the *course* for this distance.

The method of accomplishing this object will best be seen by means of an example.

Suppose a ship to start from *A* and sail to *B*, then from *B* to *C*, and then from *C* to *D*. It is required to find her position at *D*, that is, to find the difference of latitude and the departure made in going from *A* to *D*. These quantities being found, the distance *AD*, and the course *DAk*, can be calculated (Art. 28).



(REMARK.—The distances AB, BC, and CD are all supposed, in traverse sailing, to be short distances, and therefore are to be treated, like similar distances in plane sailing, as straight lines.)

Through B, A, and C suppose meridians pn, lk, and Ch, and through B, A, C, and D parallels of latitude Bf, me, Cp, and Dkn to be drawn.

Ak is the difference of latitude of AD, and kD is the departure of AD.

(1) $Ak = mh = Ch - Cm = Ch - (Cf + fm) = Ch - (Bp + eB)$.

Now, Ch is a north latitude, and Bp and eB are south latitudes; therefore, the difference of latitude of AD is equal to the difference between the *N.* latitude of one distance of the traverse and the sum of the *S. latitudes* of the other distances.

(2) $kD = nD - kn$;

$= nh + hD - Ae$;

$= pC + hD - Ae$;

But pC and hD are west departures, and Ae is an east departure; therefore, the departure for AD is equal to the difference between the sum of the *west* departures of two distances of the traverse and the *east* departure of the third distance.

In the case given above, the number of the parts of the traverse is only three, but, if a fourth distance on a course between N. and E. were given, a second north latitude and a second east departure would enter our figure, so that the *difference of latitude* between the *first* and *last* position of the ship would, in that case, be *equal to the difference between* the *sum of the north* latitudes and the *sum of the south* latitudes; and the *departure*, in passing directly from the first to the last position, would be *equal to the difference between the sum of the east* departures and the *sum of the west* departures. The same principle would hold true for a traverse of any number of distances greater than four. The proof would be similar to that given above for a traverse of three distances. The principle stated may, therefore, be taken as a *general* one.

Example 1. Suppose a ship sailing on a traverse makes courses and distances as follows: from A to B, E. b. S. 16 miles; from B to C, W. b. S. 30 miles; and from C to D, N. b. W. 14 miles. Required the distance from A to D and the course for that distance. Solution:

	Course.	Distance.	N.	S.	E.	W.
1	S. 78° 45′ E.	16	......	3.12	15.69	......
2	S. 78° 45′ W.	30	......	5.85	......	29.42
3	N. 11° 15′ W.	14	13.73	....	......	2.73
Sum.			13.73	8.97	15.69	32.15
			8.97			15.69

Dif. of lat. = 4.76 N.   Dep. = 16.46 W.

∴ (In preceding figure) $Ak = 4.76$, and $kD = 16.46$.

$$\text{Course} = kAD. \quad \frac{kD}{Ak} = \frac{16.46}{4.76} = \tan.\ 73^\circ\ 52'\ 15'' = \tan.\ kAD.$$

∴ *Course* = N. 73° 52′ 15″ W., or N. 73° 52′ W., as the result is generally given only to the nearest minute.

$$Dist. = AD = \frac{kD}{\sin.\ DAk} = \frac{16.46}{\sin.\ 73^\circ\ 52'} = 17.13 \text{ miles.}$$

2. A ship sails on a traverse, making the following courses and distances: S. E., 25 miles; E. S. E., 32 miles; E., 17 miles; N. b. W., 63 miles. Required the distance from her first to her last position, and the course. *Ans.* Dist. = 60.94 miles; course = N. 58° 29′ E.

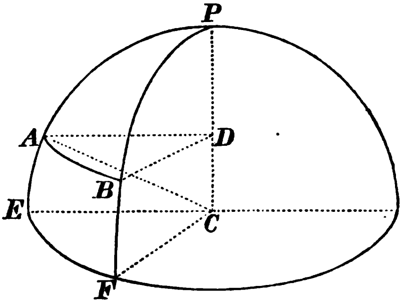
3. A ship sails on a traverse, making the following courses and distances; N. E., 25 miles; E. S. E., 40 miles; E. b. N., 35 miles; N. b. W., 33 miles. Required the course and distance from her first position to her last position. *Ans.* Course = N. 63° 16′ E.; dist. = 92.41 miles.

239. *Parallel sailing* is sailing on a parallel of latitude. In parallel sailing, therefore, a ship sails east or west ((*a*) Art. 225). The *distance* is the same as her *departure*, and the *difference* of *latitude* disappears.

The problem in parallel sailing is to convert *distance on a parallel* into *difference of longitude;* that is, given a distance between two meridians measured on a parallel, to find from it the distance between the same meridians measured on the equator ((*b*) Art. 226).

The method of solving this problem will be understood by means of the accompanying figure, which represents a part of the earth.

In this figure let C represent the centre of the earth; P be one of the poles; EF a part of the equator, CE and CF its radii; AB a part of a parallel of latitude intercepted between two meridians, PAE and PBF; and let DA and DB be the radii of this parallel.



Draw the radius AC.

$$\frac{AB}{EF} = \frac{AD}{EC} \text{ (Ch. Art. 39, V., and Art. 39, VI.).}$$

$$= \frac{AC \cos. DAC}{AC} = \cos. DAC = \cos. ACE.$$

But ACE is the latitude of A (Art. 227), or of the parallel AB.

$$\therefore \frac{\text{distance on parallel between two meridians}}{\text{distance on equator between same meridians}}$$
$$= \text{cos. lat. of parallel.}$$

Or, expressing this in other terms,

$$(1.) \quad \frac{\text{dist. on a parallel}}{\text{dif. of longitude}} = \text{cos. lat. of parallel.}$$

$$\therefore \text{(2.) dif. of longitude} = \frac{\text{dist. on parallel}}{\text{cos. lat. of parallel}}$$
$$= \text{dist. on parallel} \times \text{sec. lat.}$$

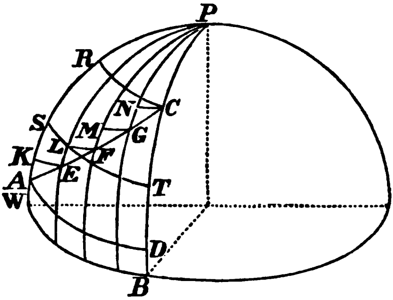
240. Since for a short distance departure is measured on a parallel of latitude ((*c*) Art. 231), in (1) of last article substituting departure for distance on a parallel, we have—

(1.) departure = dif. of long. × cos. lat.; and,

$$\text{(2.) dif. of long.} = \frac{\text{departure}}{\text{cos. lat.}} = \text{departure} \times \text{sec. lat.}$$

241. In plane sailing, when the distance sailed is short, the departure can be converted into difference of longitude by formula (1) of the preceding article, or, when the difference of longitude is given, it can be changed into departure by formula (2); in both cases the parallel of latitude being supposed to be known. But, if the distance is not short, there is danger of error, since the latitude varies from point to point of the distance, and the departure is neither the distance on a parallel through the point from which the ship sails, nor on a parallel through the point arrived at. This will be understood from the figure on page 272.

Suppose the figure to represent a part of the earth's surface, and that AC represents the distance sailed by a ship. The departure for that distance would be $KE + LF + MG$, etc. ((*c*) Art. 231), which is, evidently, equal to neither AD nor RC, since, as the meridians PE, PF, etc., meet at the pole P, the distance between them measured on a parallel diminishes as we proceed from the equator. The total departure is consequently *less* than AD and *greater* than RC. It would, also, be incorrect to convert this departure into difference of longitude by using the latitude of RC or the latitude of AD, as we really ought to use the latitude of the part departure, KE, for KE, the latitude of LF for LF, etc., and then take the sum of the differences of longitude corresponding to these departures for the whole difference of longitude. If this method were practicable, and we could make the distances AE, EF, etc., small enough, we should find the difference of longitude without appreciable error. As this method is not practicable, two other methods are used for changing departure into difference of longitude. One is the method of *middle latitude* sailing, the other is the method of *Mercator's sailing.*



242. In *middle latitude* sailing, departure is converted into difference of longitude by using, in (2) Art. 240, the latitude, whose parallel is midway between the parallel of the point sailed from and the parallel of the point arrived at.

This latitude is equal to the *half sum* of the latitude sailed from and the latitude arrived at, if both latitudes are on *the same side* of the equator, but to the half *difference*, if *one* is *north* and the *other south* of the equator.

Thus, suppose ST is the parallel midway between RC and AD; that is, suppose $AS = SR$, A and C being both north of the equator. WS is the measure of the latitude of the parallel ST (Art. 227).

$$WS = \frac{2(WA + AS)}{2} = \frac{WA + WR}{2}.$$

In a similar manner it may be shown that, in case one place is north and the other south of the equator, the middle latitude is half the difference of the latitudes of the two places.

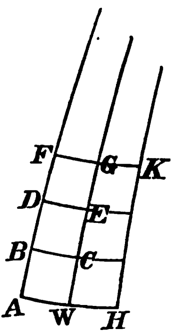
The method of middle latitude sailing is not perfectly exact, but is made nearly so by applying corrections taken from a table prepared for that purpose. For short distances or for sailing near the equator it is practically correct.

243. In *Mercator's sailing* departure is converted into difference of longitude by means of the principles of Mercator's chart.

As the meridians all pass through the poles, a chart, in order to represent correctly the earth's surface, should make the meridians' lines curved and approaching one another toward either pole. The parallels of latitude being circles smaller and smaller the nearer they are to the poles, should, on a correct chart, be shorter and shorter curves the farther they are from the equator (Ch. Art. 30, VIII.).

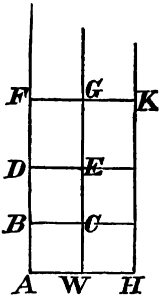
On Mercator's chart, the equator, the meridians, and parallels of latitude are all represented as *straight* lines. Meridian lines are all drawn at right angles to the equator, and are, therefore, parallel to each other. Parallels of latitude are made parallel to the equator, and therefore, like parts of any parallel, are equal to like parts of the equator (Euc., 34, I., Ch. Art. 104, I.). On Mercator's chart, therefore, the east and west dimensions of any part of the earth's surface are made too large, except near the equator. To preserve the true proportion existing between the dimensions of any particular part of the earth, the north and south dimensions are lengthened in proportion to the lengthening of the east and west dimensions. The method of accomplishing this will be understood by means of the accompanying figures.

On a chart representing the curved surface of the earth, let *FA*, *GW*, *KH*, be meridian lines; *AWH* be part of the equator; and *BC*, *DE*, and *FGK* be parts of parallels of latitude. If *AW* be a part of the equator, and *AB*, *BD*, and *DF* be like parts of a meridian, *AW*, *AB*, *BD*, and *DF* will all be equal, since the equator and a meridian are equal circles (Arts. 223 and 224, Ch. Art. 29, VIII.). *BC*, *DE*, and *FG* will be the same parts of the parallels of latitude to which they belong, that *AW*, *AB*, *BD*, and *DF* are of their circles, since any one of them, as *BC*, measures the same plane angle which *AW* measures; that is, the plane angle of the diedral angle of the planes of the meridians (Ch. Art. 39, VI., and Art. 39, V.).



Now $AB = AW = BC$ sec. lat. of BC ((2) Art. 239);
$BD = AW = DE$ sec. lat. of DE;
$DF = AW = FG$ sec. lat. of FG.

On Mercator's chart, *BC* being made equal to *AW*, its length is equal to its length, on a chart representing a curved surface, multiplied by the secant of its latitude; so, also, *DE* on Mercator's chart is equal to *its* length, on a chart representing a curved surface, multiplied by the secant of its latitude. To make the north and south dimension lengthen in the same proportion with the lengthening of the east and west dimension, *AB* on Mercator's chart is equal to its length, on a chart representing a curved surface, multiplied by the secant of latitude of



BC; *BD* is equal to its length multiplied by secant of latitude of *DE*, etc., etc. Mercator's chart may, therefore, be said to be a chart of the earth's surface on which the unit of the scale of representation is continually changing for each latitude. Near the equator the parts of the earth's surface are correctly represented. As we go north or south from the equator, the parts of the earth are enlarged as compared with the parts near the equator.

Thus, in latitude 60° north or south, the east and west dimension of a part of the earth compared with its true dimension, as represented in terms of the unit of the scale of representation at the equator, is equal to the true dimension multiplied by secant 60°; but sec. 60° is 2 (Art. 16), consequently the east and west dimension is doubled. As the north and south dimension is also doubled, the area is quadrupled.

244. The path of a ship continuing on the same course, is on Mercator's chart, a *straight* line, since to continue on the same course the ship must cut each of the meridians at the same angle, and the meridians are parallel straight lines ((*b*) Art. 231, Euc. 29, I., Ch. Art. 51, I.).

245. A table of *meridional parts* is a table of the lengths of any required number of minutes or degrees of Mercator's meridian (proceeding from the equator), in terms of some small part of the true meridian taken as a unit. The unit generally taken is 1′.

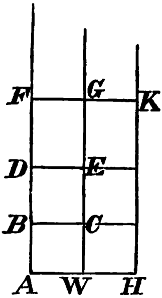
As Mercator's meridian is longer than the true meridian on a chart representing a curved surface, and is continually lengthening, the *number of parts* in a certain number of degrees and minutes of the table, will generally be greater than the number of *minutes* in the corresponding number of degrees and minutes of true meridian.

Thus, for example, the number of parts of 8° of the table of meridional parts is 481.6, while the number of minutes of 8° of true meridian is 480.

Near the equator the number of minutes of true meridian, and the number of meridional parts of the same degree measure, are equal.

This will be seen by referring to the method of constructing Mercator's chart.

Taking the second figure of Art. 243, let AW represent 1′ of the equator, and AB, BD, DF be each 1′ of meridian. Then BC, DE, FG are respectively 1′ of parallel of 1′, 1′ of parallel of 2′, 1′ of parallel of 3′.



But on Mercator's chart,

$$AB = BC \times \text{sec. lat. of } BC;$$
$$= AW \times \text{sec. } 1';$$

also, $BD = DE \times \text{sec. lat. } DE;$

$$= AW \times \text{sec. } 2';$$

and $DF = AW \times \text{sec. } 3'$, etc., etc. (Art. 243).

$$\therefore AF = AB + BD + DF$$
$$= AW \times (\text{sec. } 1' + \text{sec. } 2' + \text{sec. } 3').$$

But as AW was taken as our unit,

$$\therefore AF = \text{sec. } 1' + \text{sec. } 2' + \text{sec. } 3'.$$

As AF is only 3′ of Mercator's meridian, and as sec. 1′, sec. 2′, sec. 3′ are each 1, AF would equal 3; that is, the number of *parts* of 3′ of Mercator's meridian would be the same as the number of minutes of true meridian. But the length of 40° of Mercator's meridian could be shown to be AW(sec. 1′ + sec. 2′ + sec. 3′ +, etc., up to sec. 40°); or, as AW is 1, to be the sum of the secants of all the angles, differing by 1′, from 1′ to 40° inclusive, a sum which by inspection can be seen to be greater than 2400—the number of minutes in 40°.

(*a.*) *Meridional difference of latitude* is the distance on Mercator's meridian between two parallels of latitude.

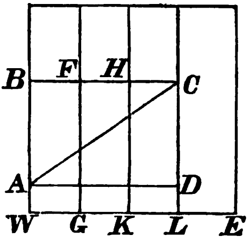
Where the latitudes of two places are given the *meridional difference* of *latitude* is found by taking the *meridional parts* of the less latitude *from* the *meridi-*

onal parts of the greater, if *both* are *north*, or *both* are *south* latitudes; but, by *adding* the meridional parts of the two latitudes, if *one* is *north* and the *other south* latitude.

The rule is the same as for finding the true difference of latitude (Art. 228), except that *meridional parts* of latitude are used instead of latitude.

EXAMPLE 1. Lat. of Newport, R. I., is	41° 29′ N.	Merid. parts	2740.2
Lat. of Savannah, Ga., is	32° 5′ N.	Merid. parts	2034.3
Dif. of lat.	9° 24′	Merid. dif. lat.	705.9
2. Lat. of Cape Nassau is	7° 32′ N.	Merid. parts	453.3
Lat. of Cape St. Roque is	5° 28′ S.	Merid. parts	328.5
Dif. of lat.	13° 0′	Merid. dif. lat.	781.8

If one latitude is given, and the meridional difference of latitude is found, the latitude required is found by *adding* the meridional parts of the given latitude to the meridional difference of latitude, if the place whose latitude is required is farther from the equator than the place whose latitude is given, and if both places are on the same side of the equator; but, by *subtracting* the meridional difference of latitude from the meridional parts of the given latitude if the place whose latitude required is nearer the equator than the place whose latitude is given; the degrees and minutes, answering to the result as found in the table of meridional parts, will be the latitude required.



This will be evident from the figure, in which WL represents the equator on Mercator's chart.

If the latitude of A is given, the meridional parts, or the distance AW, can be found from the table. AB being the meridional difference of latitude, the meridional parts of C or the distance, $LC = WB = WA + AB$.

If the latitude of C is given, then from the table CL ($= WB$) is found; then, $AW = WB - AB = CL - AB$.

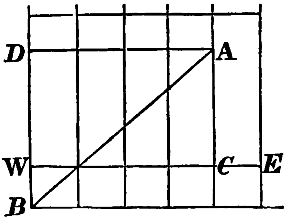
EXAMPLE 1. Lat. of place left is 25° 6′ N. Merid. parts = 1556.6
Ship sails northerly till she makes merid. dif. of lat. 750.
Meridional parts of place arrived at = 2306.6
Therefore, from table, latitude arrived at is (nearly) 35° 51′ N.

2. Latitude of place left is 46° 10′ S. Merid. parts = 3130.
Ship going northerly, her merid. dif. of lat. is found to be = 825.6
Merid. parts of lat. arrived at = 2304.4
Therefore, latitude arrived at is 35° 49′ S.

If a ship starting from one side of the equator sails to a point on the other side, the latitude of the point arrived at is found by subtracting the meridional parts of the given latitude from the meridional difference of latitude; the result will be the meridional parts of the required latitude.

EXAMPLE. The meridional difference of latitude is.......,.. 1805.8
which is made by a ship going *north*, starting from lat. 8° 41′ S.. 523.
Therefore, latitude arrived at is 20° 54′ N. Merid. parts..... 1282.8

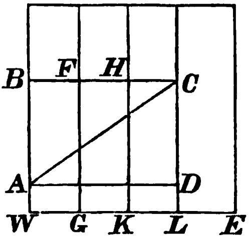
It is evident, therefore, if the meridional difference of latitude made by a ship sailing *from* a point on either side of the equator *toward* a point on the opposite side, exceeds the meridional parts of the latitude left, that the ship has crossed the line and has arrived at a N. latitude, if the latitude left was S., but has arrived at a S. latitude if the latitude left was N.



Thus, on the figure, WE representing the equator, a ship sails from B toward A. BW represents the meridional parts of the latitude left, BD is the meridional difference of latitude, AC represents the meridional parts of the latitude arrived at. $AC = WD = BD - BW$.

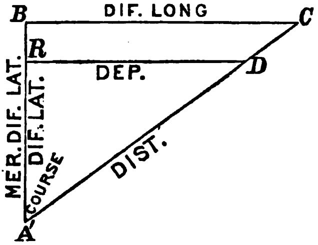
246. *Combining Mercator's sailing with plane sailing.*

Let the figure represent a part of Mercator's chart, on which WE is the equator, and AC is the distance between two points A and C. CAB is the course for that distance. If, from C, CB be drawn perpendicular to the meridian WB, AB will be the meridional difference of latitude, and BC will be the departure ((*c*) Art. 231).



Now, $BC = WL$ (Euc., 34, I., Ch. Art. 104, I.); that is, *departure*, on Mercator's chart, *equals difference of longitude* ((*b*) Art. 226).

If, in plane sailing, the same course and distance were represented by the hypotenuse and acute angle of a right-angled triangle, $A'RD$ (Art. 232), $A'R$ would be true difference of latitude, RD would be departure.



Now, ABC and $A'RD$ are similar triangles (Ch. Art. 26, III.), since the angles A and A' are equal, as they both represent the same course, and the angles R and B are right angles. Placing the angle A upon A', the two triangles may be combined, as in the figure; then—

$$\frac{A'R}{A'B}=\frac{RD}{BC};$$ that is,

(1.) $$\frac{\text{dif. of lat.}}{\text{merid. dif. of lat.}}=\frac{\text{dep.}}{\text{dif. of long.}}$$

Also, $BC = A'B \times \tan. A'$; that is,

(2.) dif. of long. = merid. dif. of lat. × tan. course.

By means of these two triangles all cases of Mercator's sailing may be solved, and the position of a ship at sea may be determined from the usual data.

The latitude of one position of the ship, either of the point left or the point arrived at, must always be known in order to use Mercator's sailing.

The line $A'C$ is not required in calculations. $A'D$ represents the true distance.

EXAMPLE 1. A ship starting from lat. 37° N., long. 10° W., sails on a course between N. and E. to lat. 41° N., making a distance of 300 miles. Required the course and the longitude arrived at.

Lat. 41° N.	Merid. parts = 2701.6
Lat. 37° N.	Merid. parts = 2392.6
Dif. of lat. = 4° = 240′	Merid. dif. of lat. = 309.

Taking the figure of the preceding article, $A'D = 300$, $A'R = 240$, and $A'B = 309$.

$$\frac{A'R}{A'D} = \frac{240}{300} = \cos. A' = \cos. \text{course}. \quad BC = A'B \times \tan. A'$$

		dif. long. = 309 × tan. 36° 52′ 12″	
Log. 240 =	2.380211	Log. 309 =	2.489958
Log. 300 =	2.477121	Log. tan. 36° 52′ 12″ =	9.875063
Log. cos. 36° 52′ 12″	9.903090	Log. 231.75	2.365021
Course = N. 36° 52′ E.	93	dif. long. = 3° 51.75′ E.	4926
	3 cor. 2″	cor. 5	95

Long. left, 10° W.
Dif. of long. 3° 51.75′ E.

Long. arrived at = 6° 8.25′ W.

2. A ship, leaving lat. 50° 10′ N., long. 60° E., sails E. S. E. till her departure is 957 miles. Required latitude and longitude arrived at, and the distance sailed.

$$\frac{957}{\sin. 67° 30'} = \text{dist.}^{*} \qquad \frac{957}{\tan. 67° 30'} = \text{dif. of lat.}$$

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* No figure is given for this example, but the student is advised to plot the figure for it, and the figure for each of the examples which follow.

Log. 957 =	2.980912	Log. 957 =	2.980912
Log. sin. 67° 30 =	9.965615	Log. tan. 67° 30′ =	10.382776
Dist. = 1035.85 miles =	3.015297	Log. 396.4	2.598136
	4940	Dif. lat. = 6° 36.4′ S.	4
cor. 85	357	Lat. left. 50° 10′ N.	

Lat. arrived at 43° 33.6′ N.

Merid. parts of 50° 10′ = 3490.1
Merid. parts of 43° 33.6′ = 2909.2

		Merid. dif. of lat. = 580.9 log. =	2.764101
		Log. tan. 67° 30′ =	10.382776
Dif. of long. =	23° 22.4′ E.	Log. 1402.4 =	3.146877
Long. left,	60° E.		748
Long. arrived at	83° 22.4′ E.	cor. 4	129

3. A ship, leaving lat. 49° 57′ N., long. 5° 14′ W., sails on a course S. 39° W. till her latitude is found to be 45° 31′ N. Required the distance sailed and the longitude arrived at.

Ans. Dist. = 342.28 miles; long. = 10° 34.5′ W.

4. A ship sails from lat. 49° 57′ N., long. 5° 14′ W., to lat. 39° 20′ N., when her departure is found to be 789 miles. Required course and distance sailed, and longitude arrived at.

Ans. Course S. 51° 5′ W.; dist. = 1014 miles; long. = 23° 47.6′ W.

5. A ship, leaving port in lat. 14° 45′ N., long. 17° 38′ W., sails on a course S. 28° 7¼′ W., till her longitude is 29° 26′ W. Required the distance sailed and the latitude arrived at.

Ans. Dist. = 1500.1 miles; lat. 7° 18′ S.

6. Required the course and distance from port of San Francisco, lat. 37° 48′ N., long. 122° 21′ W., to Jeddo, lat. 35° 40′ N., long. 140° E.

Ans. Course S. 88° 26′ 16″ W.; dist. = 4695 miles.

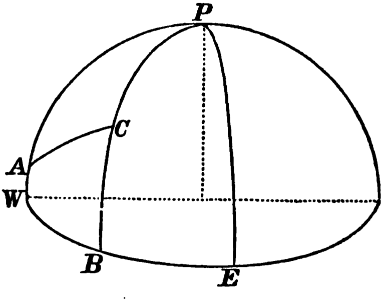
CHAPTER XX.

SOME APPLICATIONS OF SPHERICAL TRIGONOMETRY.

ART. 247. *To find the shortest distance between two points on the earth, the latitude and longitude of each point being given.*

We have to find the length of the arc of a great circle not greater than a semi-circumference joining the two points (Ch. Art. 105, VIII.).

Suppose A and C to represent the two points. If P represents the pole of the earth, WE a part of the



the equator, PE the meridian from which longitude is reckoned, and PW and PB meridians through A and C; then, WB will be the difference of longitiude of A and C ((*b*) Art. 226); WA will measure the latitude of A, and BC will measure the latitude of C (Art. 227).

In the spherical triangle APC,
$AP = PW - AW = 90° -$ lat. of A;
$PC = PB - BC = 90° -$ lat. of C;
angle APC is measured by arc WB (Ch. Art. 57, VIII.), or, degrees of $APC =$ degrees of difference of longitude; therefore, we have given two sides and included angle of a spherical triangle to find the third side. The third side can be found by Art. 157.

248. A better method of obtaining the third side is to use one of the formulas of Art. 145.

Thus, denote the sides opposite A, P, and C by a, p, and c respectively.

In formula (b) of Art. 145, substituting P for B, and p for b, we have

$$\cos. P = \frac{\cos. p - \cos. c \cos. a}{\sin. c \sin. a}, \text{ therefore}$$

$$(1) \quad \cos. p = \cos. c \cos. a + \cos. P \sin. c \sin. a,$$

an equation which will give the value of P without ambiguity (Arts. 46 and 47).

(1) can be written in the form

$$\cos. p = \cos. a \left(\cos. c + \cos. P \sin. c \frac{\sin. a}{\cos. a}\right).$$

In this expression assume $\tan. x = \cos. P \dfrac{\sin. a}{\cos. a}$, or

$$(2) \quad \tan. x = \cos. P \tan. a \text{ (Art. 58); then}$$

$$\cos. p = \cos. a \left(\cos. c + \sin. c \frac{\sin. x}{\cos. x}\right),$$

$$= \frac{\cos. a (\cos. c \cos. x + \sin. c \sin. x)}{\cos. x}$$

$$\therefore (3) \quad \cos. p = \frac{\cos. a \cos. (c - x)}{\cos. x} \text{ ((}b\text{) Art. 68).}$$

(3) is more convenient for use of logarithms than (1).

To find p by (3) we first find x from (2), and then substitute its value in (3).

EXAMPLE 1. Required the shortest distance between a place in lat. 17° 58′ N., long. 76° 49′ W., and Limerick, Ireland, in lat. 52° 42′ N., long. 9° 12′ W.

$PC = a = 90° - 52° 42' = 37° 18'$. $P = 76° 49' - 9° 12' = 67° 37'$

$PA = c = 90° - 17° 58' = 72° 2'$.

$$\begin{aligned}
\tan. x &= \cos. P \tan a. \\
&= \cos. 67° 37' \tan. 37° 18' \\
\text{Log. cos. } 67° 37' &= 9.580699 \\
\text{Log. tan. } 37° 18' &= 9.881839 \\
\hline
\text{Log. tan. } 16° 10' 37.6'' &= 9.462538 \\
x &= 16° 10' 37.6'' \\
c - x &= 72° 2' - (16° 10' 37.6'') \\
&= 55° 51' 22.4''
\end{aligned}$$

$$\begin{aligned}
\cos. p &= \frac{\cos. a \cos. (c - x)}{\cos. x} \\
&= \frac{\cos. 37° 18' \cos. 55° 51' 22.4''}{\cos. 16° 10' 37.6''} \\
\text{Log. cos. } 37° 18' &= 9.900626 \\
\text{Log. cos. } 56° 51' 22.4'' &= 9.749173 \\
\text{Ar.co.log. cos. } 16° 10' 37.6'' &= 0.017546 \\
\hline
\text{Log. cos. } 62° 17' 50'' &= 9.667345 \\
\therefore p &= 62° 17' 50'' \\
&= 3737.8 \text{ nautical miles.}
\end{aligned}$$

2. Required the shortest distance between Porto Rico, in lat. 18° 29′ N., long. 66° 9′ W., and Amsterdam, in lat. 52° 22′ N., long. 4° 51′ E.

Ans. 3835.28 miles.

248. When a ship sails between two points, making the shortest distance, she sails on the arc of a great circle. In order to do this she cannot continue on the *same* course, as the arc of the circle does not cut all the meridians at the same angle, unless the arc sailed on be the equator (Ch. Art. 57, VIII.).

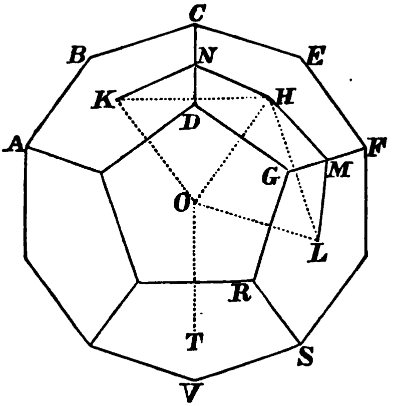
249. *To find the volumes of the regular polyedrons.*

(*a*) Let the figure represent a regular polyedron (a dodecaedron).

From the centres K, H, and L, of three faces draw the apothems KN, HN, HM, and ML. Draw the straight lines KH and HL. The planes KNH, HML, are perpendicular to the lines CD and GF respectively (Ch. Art. 13, VI.). The plane KNH is perpendicular to the faces $ABCD$, $CDGF$, and the plane HML is perpendicular to the faces $CDGF$, $GRSF$ (Ch. Art. 47, VI.).

From *K* draw a perpendicular to the face *ADCB*, and from *H* draw a perpendicular to the face *CDGF*. These lines will both lie in the plane *KNH* (Ch. Art. 50, VI.), and will meet; otherwise, the faces *ABCD* and *CDGF* will be one plane (Ch. Art. 34, VI.). *OH* also lies in the plane *HML* (Ch. Art. 50, VI.). Let these lines meet in *O*, and from *O* draw *OL*.

The angles *KNH* and *HML* are the plane angles of the diedral angles *CD* and *GF* (Ch. Art. 39, VI.), and are, therefore, equal, since the polyedron is regular, and each diedral angle is equal to any other diedral angle. Also, the lines *KN*, *NH*, *HM*, and *ML* are equal, being apothems of equal polygons. Consequently, the triangles *KNH* and *HML* are equal isosceles triangles, and *KH* is equal to *HL*, and the angles *NKH*, *NHK*, *MHL*, and *MLH* are all equal.



The angles *OHN* and *OHM* are right angles; therefore, taking equals from equals, the angles *OHK* and *OHL* are equal. Consequently, *OKH* and *OHL* are equal triangles (Euc., 4, I., Ch. 20, I.), and *OK* is equal to *OL*, and the angle *OKH* is equal to the angle *OLH*. Adding equals to equals, the angle *OKN* is equal to the angle *OLM*; but the angle *OKM* is a right angle, therefore the angle *OLM* is a right angle, and *OL* is perpendicular to the face *FGRS* at its middle point (Ch. Art. 49, VI.).

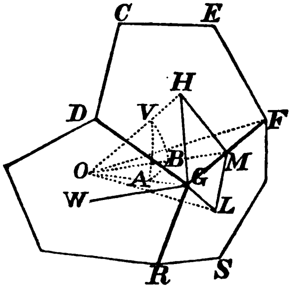
OKN and *OHN* are equal, being right angles, and *NKH* and *NHK* are equal, being angles at the base of an isoceles triangle; therefore *OKH* and *OHK* are equal angles, and the sides *OK* and *OH* are equal; but *OK* was proved equal to *OL*; therefore *OK*, *OH*, and *OL* are equal to each other, and the faces $ABCD$, $CDGF$, and $GRSF$ are all equally distant from the point O.

In the same manner it may be proved that the lines from O to the centres of the other faces of the polyedron are perpendicular to those faces respectively, and are equal to each other and to the lines OK, OH, and OL. If, therefore, with O as a centre, and one of these lines as a radius, a sphere be described, it will touch all the faces of the polyedron (Ch. Art. 43, VIII.), and will be inscribed in the polyedron.

If the polyedron be divided into pyramids, having their vertices at O, and their bases the faces of the polyedron, *these pyramids will all be equal* (Ch. Art. 55, VII.), and *their common altitude will be the radius of the sphere* inscribed in the polyedron.

(*b*) Let the figure represent one of the polyedral angles, G, of the polyedron, DGF, DGR, and RFG being the equal face angles which form it. If W, H, and L be the centres of the faces, and the lines WG, HG, and LG be drawn, the face angles will be bisected (Ch. Art. 8, V.).

If planes be passed through these lines, and O the centre of the inscribed sphere, a number of equal diedral angles will be formed on OG, each of which is equal to the diedral angle $MGOH$. As the sum of these angles equals four right angles, or, in degrees, 360°, any one of them, as $HGOM$, will be equal to 360° divided by *twice the number* of the *faces* which form the polyedral angle. Denoting by m the number of the faces at G, $HGOM = \frac{360°}{2m} = \frac{180°}{m}$.



(*c*) If planes were passed through HO, and each of the vertices of the face $CDGFE$, there would be formed about HO a number of equal diedral angles, the measure of each of which would be $\frac{360°}{n}$, if n denotes the number of the *sides* of the regular polygons, which are the faces of the polyedron. The diedral angle $GOHM$ would be one-half of one of these equal angles, and its measure would be $\frac{360°}{2n} = \frac{180°}{n}$.

The two triangles HMO and LMO are equal, since they have three sides of one equal to three sides of the other, each to each; therefore, the angle HML is bisected by the straight line OM. HML measures the inclination of any two adjacent faces of the polyedron, and HMO is half of that angle.

(*d*) From O as centre suppose a sphere to be described with radius OA. Then VBA would be a right-angled spherical triangle (Art. 104), for the plane $HMLO$ is perpendicular to the line GF, and therefore to the plane GOF.

By Napier's rule, cos. $A =$ sin. V cos. BV (Art. 115);

but $A =$ the diedral angle $HGOM = \frac{180°}{m}$;

$V =$ the diedral angle $GOHM = \frac{180°}{n}$;

BV measures the angle HOM, which equals $90° - HMO$;
therefore, cos. $BV =$ cos. $(90° - HMO) =$ sin. HMO (Art. 5);
substituting in first equation—

$$\therefore \cos. \frac{180°}{m} = \sin. \frac{180°}{n} \times \sin. HMO, \text{ therefore}$$

$$(e)\ \sin. HMO = \frac{\cos. \frac{180°}{m}}{\sin. \frac{180°}{n}}$$

$$HM = GM \cot. GHM\ ((4) \text{ Art. } 31) = GM \cot. \frac{180°}{n}.$$

$$(f)\ OH = HM \tan. HMO = GM \cot. \frac{180°}{n} \tan. HMO.$$

By (*e*) the angle HMO can be found, and by (*f*) the line OH, which is the radius of the sphere inscribed in the polyedron.

The *surface* of the polyedron can be found by Art. 171, since the surface is composed of regular polygons.

The *volume* of the polyedron is equal to the sum of the pyramids, having their vertices at O; that is, is equal to $\frac{1}{3}$ $OH \times$ the surface of the polyedron (Ch. Art. 54, VII.).

Example 1. The lateral edge of a regular dodecaedron is 4 inches. Required its volume.

$m = 3$; $n = 5$; $GF = 4$.

$$\text{Sin. } HMO = \frac{\cos. \frac{180^\circ}{3}}{\sin. \frac{180^\circ}{5}} = \frac{\cos. 60^\circ}{\sin. 36^\circ}$$

$$\begin{aligned} \text{Log. cos. } 60^\circ &= 9.698970 \\ \text{Log. sin. } 36^\circ &= 9.769219 \\ \hline \text{Log. sin. } 58^\circ\ 16'\ 57'' &= 9.929751 \end{aligned}$$

$\therefore HMO = 58^\circ\ 16'\ 57''$

$$\begin{aligned} OH &= HM \tan. HMO \\ &= GM \cot. GHM \tan. HMO. \\ &= 2 \cot. 36^\circ \tan. 58^\circ\ 16'\ 57'' \end{aligned}$$

$$\text{Area of surface} = 12 \times 4^2 \times \frac{5}{4} \cot. 36^\circ$$

$$\begin{aligned} \text{Vol.} &= \tfrac{1}{3} \times 2 \cot. 36^\circ \tan. 58^\circ\ 16'\ 57'' \\ &\quad \times 12 \times \frac{4^2}{4} \times 5 \cot. 36^\circ \\ &= 160 \cot.^2 36^\circ \tan. 58^\circ\ 16'\ 57'' \end{aligned}$$

$$\begin{aligned} \text{Log. } 160 &= 2.204120 \\ 2 \text{ log. cot. } 36^\circ &= 20.277478 \\ \text{Log. tan. } 58^\circ\ 16'\ 57'' &= 10.208987 \\ \hline \text{Log. } 490.44 &= 2.690585 \end{aligned}$$

Vol. = 490.44 cu. in.

2. What is the angle of inclination of two adjacent faces of a regular octaedron? Required the volume if the edge is 1 inch.

Ans. (by logarithms). Angle = 109° 28′ 16″. Vol. = 0.4714 cub. in.

3. Required the angle of inclination of two adjacent faces of a regular tetraedron. What is the volume if an edge is 10 inches?

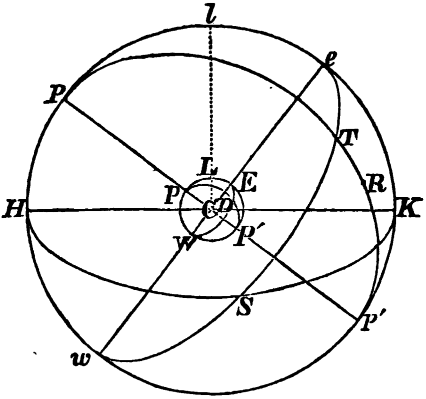
Ans. Angle = 70° 31′ 43.4″. Vol. = 117.85 cub. in.

4. Required the angle of inclination of two adjacent faces of a regular icosaedron. Required also the volume when the edge is 1 inch.

Ans. Angle = 138° 11′ 22.6″. Vol. = 2.1817 cub. in. (nearly).

250. Let the figure represent the earth, $PWAP'E$, surrounded by the celestial sphere.

(a) If the axis PP' be produced to meet the surface of this (imaginary) sphere, the points p and p' at which the axis meets it are called the *celestial poles*, and the line *pp′* is called the *axis* of the celestial sphere.



(*b*) The plane of the equator, *WDE*, produced, intersects the celestial sphere in the *celestial equator*, *wSe*.

(*c*) The planes of the meridians *PEP′*, *PDP′*, intersect the celestial sphere in great circles *pep′*, *pTp′*, which are called *hour circles*.

Since the earth revolves upon its axis once in twenty-four hours, every point on a celestial meridian would appear, to an observer on the earth's surface, to move through a complete circumference, or 360°, during that time. If, now, the celestial meridians are drawn at intervals of 15° (on the equator), there will be 24 such meridians. Since the time in which *all* these meridians pass by an observer is 24 hours, the interval of time of passage between two successive meridians will be one hour, since 24 of them pass by him in 24 hours. If meridians are drawn at intervals of 1°, the interval of time of passage of two such meridians will be $\frac{60 \text{ minutes}}{15}$, or 4 minutes. Thus, the passage of these meridians or of points on them being measured by time or degrees, we can convert one measure into the other.

(*d*) The angles made by these meridians at p and p' are called hour-angles, and these angles are measured by the arcs which they intercept on the arc of the equator wSe (Ch. Art. 57, VIII.).

(*e*) The *celestial horizon* of any place, on the earth's surface, is the circle made by a plane passing through the centre of the earth parallel to the plane of the horizon at that place, and intersecting the celestial sphere.

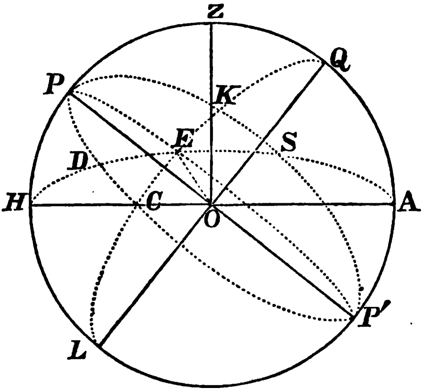
The celestial horizon of the point L is HSK.

(*f*) If a straight line be drawn from the centre, C, to L, and this line be produced through L to meet the celestial sphere at l, l will be the zenith of L; lp will measure the *zenith distance* of L, and le will measure the *celestial latitude* of L. The zenith distance is the complement of the celestial latitude. The *degree measure* of the *celestial* latitude is the same as that of the *terrestrial latitude*, since they both subtend the same angle at the centre of the earth. Thus, le and LE both subtend the angle LCE.

(*g*) The *declination* of an object in the heavens is its distance from the celestial equator measured in degrees, minutes, and seconds of a celestial meridian, and is + if north but − if south of the equator.

Thus, in the preceding figure, TR is the south declination of the point R.

(*h*) The path in the heavens in which the sun, to an observer on the earth, appears to move during a year, is nearly a great circle, and is called the *ecliptic*. The plane of this circle is inclined to the plane of the celestial equator at an angle of 23° 27′ (nearly). The two points in which these circles intersect are called *equinoctial* points, or simply the *equinoxes*. At these points, the sun being *on* the equator, its declination is 0°. At the two points, each 90° (on the ecliptic) from the *equinoxes*, its declination is 23° 27′ (Ch. Art. 57, VIII.). At one of these points it has a N. declination, and at the other a S. declination. At intermediate points its declination varies from 0° to 23° 27′, and is N. or S. according as it is in the part of the ecliptic which is N. or S. of the equator.



251. *To find the time of sunrise or sunset* for a given day *at any place on the earth*, the *latitude* of the place and the *sun's declination* for the day *being given*.

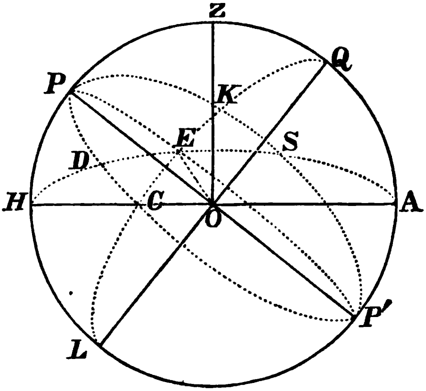
(*a*) Let the figure represent a half of the celestial sphere, the observer being at *O*, and looking on the concave surface of the celestial sphere.

Suppose *P* and *P′* to be the celestial poles.

Let *HEA* be the horizon of the point *O*.

Suppose the plane which bisects the sphere to pass through the zenith, *Z*, of the point *O*. If from *O* a perpendicular be drawn to this bisecting plane, and if the point *E* where this line insersects the celestial sphere be the east point of the horizon, then *E* will be at a quadrant's distance from each point of the circle *HPZQA* (Ch. Arts. 27 and 37, VIII.). *E* is 6 hours, or 90°, from *Z* ((*c*) Art. 250).

Through *O* draw *LQ*, the diameter of the equator. *PO* is perpendicular to *OE* and *OQ*, and therefore to their plane (Ch. Art. 13, VI.). Therefore, the plane of the equator will be this plane, and the arc of the equator, *LEQ*, will pass through *E*, the east point of the horizon.



Let *PEP′* be the six o'clock hour-circle; that is, the meridian which is on the horizon at six o'clock.

Let *D* be the position of the sun at its rising, when its declination, DC, is north; let S be its position when its declination, KS, is south; and let $PDCP'$, and $PKSP'$ be hour-circles through these two positions. The angles at K and C are right angles (Ch. Art. 58, VIII.), and CD and KS measure declination ((*g*) Art. 250).

The angles DEC and KES are equal, and are each measured by LH (Ch. Art. 57, VIII.), or AQ which is $AZ-QZ$, or the complement of the latitude ((*f*) Art. 250).

In the right-angled triangle DEC we have DC and DEC given to find CE; or, in SEK, SK and SEK are given to find EK.

Sin. $CE =$ cot. E tan. CD (Art. 115)

= tan. lat. × tan. declination.

CE and EK both measure, in degrees, the distance of the hour-circle on which the sun appears on the horizon from the six o'clock hour-circle. Converting the degree measure into hours and minutes, we subtract the result from 6 hours for the time of sunrise if the given latitude and given declination are *both north or both south*, but add the result to 6 hours if *one* is *north* and the *other south.*

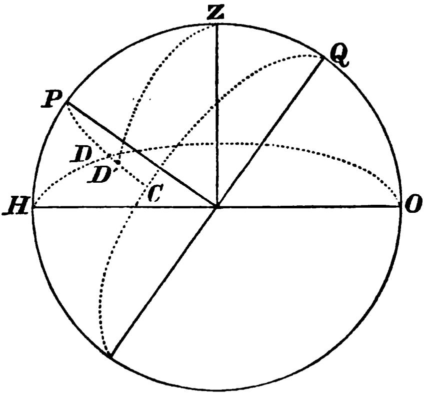
Thus, $LC = LE - CE$ when the sun is at D, but $LK = LE + EK$ when the sun is at S.

The solution of the problem has been given only for the time of sunrise. The solution is similar for the time of sunset, and the result is the same; but the rule for applying the result to 6 hours is the reverse of that for sunrise; that is, the result is to be *added* to 6 hours, if the given latitude and the given declination are both north or both south, but to be *subtracted* from 6 hours, if *one* is *north* and the *other south.*

This method allows nothing for the effect of refraction, by which the sun is made to *appear* above the horizon, while it is really below it. It is a method, also, which gives only the time of appearance of the sun's *centre* on the horizon. The following method gives a result in which allowance is made for refraction, and which gives, also, the time of appearance of the sun's upper limb (or upper edge).

(*b*) The distance of the sun's centre to its outer rim is, in angular measurement, 16′. The allowance to be made for refraction is 34′. Now, on the figure, *ZD* is 90°; therefore the distance from *Z* to the point where the sun first becomes visible is 90° 50′. Call this point *D′*.

In the previous case we calculated the arc of the equator intercepted between two meridians. By this



method we calculate the angle, at the pole *P*, between the hour-circle on which the sun lies at the time of its appearance and the 12 o'clock or zenith circle *PZO*; that is, we calculate the hour-angle *CPQ*.

In the triangle *D′PZ*, *PZ* is known, being the complement of the latitude; PD' is also known, being the distance from P, and is $90° \mp$ the declination (the upper or lower sign being taken according as the declination is of the same name with the pole P, or of an opposite name.)

$ZD' = 90°\ 50'$.

Let $D'P = b$; $PZ = c$; and $D'Z = a$;

Let $s = \dfrac{a + b + c}{2}$;

then $\tan.\tfrac{1}{2}P = \tan.\tfrac{1}{2}A = \sqrt{\dfrac{\sin.(s - b)\sin.(s - c)}{\sin. s \sin.(s - a)}}$

(Art. 155).

Examples under this article give results only for *apparent* solar time, and in the first method without regard to refraction. To obtain results for *mean* solar time (ordinary time) a correction must be applied, due to the difference between *apparent* and *mean* time. A discussion of the principles on which corrections are applied belongs to astronomy. Nothing more is attempted here than the application of spherical trigonometry to the solution of the problem given at the beginning of the article.

EXAMPLE 1. In lat. 41° 18′ N. on a given day the sun's declination is 14° 44′ 45″ N. What is the time of the sun's rising (by the first method)?

Denoting by x the quantity to be subtracted from 6 hours;

$\text{Sin.}\,x = \tan.\,41°\ 18' \times \tan.\,41°\ 44'\ 45''$;

$\therefore x = 13°\ 22'\ 9''$;

$= 53.48$ **minutes of time.**

Time of sunrise = 6 h. − 53.48 = 5 h. 6.52 m.

2. In lat. 41° 46′ N. on a given day the sun's declination is 15° 25′ S. Required the time of sunrise (by the first method). *Ans.* 6 h. 57 m.

3. In lat. 41° 29′ N. (on a given day) the sun's declination is found to be 16° 26′ N. Required the time of sunrise (by the second method).

***Ans.* $P = 106°\ 19'\ 20''$, the hour-angle measured from the zenith or 12 o'clock hour circle. Taking this from 180°, we have 73° 40′ 40″, which gives 4 h. 54.7 m. as the time of sunrise.**

4. At Philadelphia, in lat. 39° 57′ N., on a given day the sun's declination is observed to be 13° 48′ S. Required the time of sunset. In this example, as the declination is S., the side PD' in the figure for the *second method* is 90° + 13° 48′. *Ans.* 5 h. 17.07 m.

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